D_2^α: A Family of Democratic Logics *†

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1 Introduction

This paper offers a paraconsistent logical framework for analyzing collective judgments. The framework presented is useful for clarifying the extent to which various propositions (and inferences based on those propositions) are *proportionally* true in group-discourse settings. Formally, this task is called *judgment aggregation*, which attempts to answer the following question (posed by Christian List):

> "How can a group of individuals make consistent collective judgments on a set of propositions on the basis of the group members' individual judgments on them?" [6]

Though the difficulty of judgment aggregation has been evident since the introduction of Condorcet's paradox in the 18^{th} century[1], the field remained relatively inactive until the mid-20^{th} century when Kenneth Arrow won the Nobel Prize for his seminal impossibility result. Since then (and particularly in the last 20 years), the field has exploded with research spanning multiple disciplines [6].

This paper will not consider Arrow's Theorem explicitly, but will instead focus on a more general impossibility result–the so-called *discursive dilemma* or *doctrinal paradox*[2], which presents the problem of judgment aggregation in a logical setting. This impossibility result was first formalized by List and Pettit in 2002, but has since been extended numerous times [6] [1] [2]. This paper will examine the applicability of Stanislaw Jaskowski's non-adjunctive two-valued logic, D_2, to the *discursive dilemma*. As I will show, though D_2 produces "consistent" aggregation results, its formulation loses or ignores practical information about the judgment aggregation problem under investigation. In light of this, I propose an extension of D_2 that retains maximal inferential

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[1] First presented in 1785

[2] For a detailed discussion of the relationship between Arrow's Theorem and the discursive dilemma, see List and Pettit 2004 [7]

information in the judgment aggregation setting without producing "inconsistent" results. To achieve this, I introduce a new modal operator: $\diamond_{i/d}$ [3], defined as a unary operator such that:

$$\diamond_{i/d} p \equiv p \textit{ is held true in } i \textit{ out of } d \textit{ worlds} \tag{1}$$

Using this operator, I define a family of *democratic* logics, nearly identical in structure to D_2, called D_2^α. I say a *family* of logics because α represents a parameter whose value defines a specific logic. That is for a given choice of α, say 2/3, each thesis in the logic $D_2^{(2/3)}$ becomes a thesis of the two-valued logic modal M_2 when preceded with the operator $\diamond_{2/3}$. Using Jaskowski's 1948 paper [4] [5] as a guide, I will examine a number of D_2 theses and non-theses, comparing them with various D_2^α counterparts. Through this discussion, I hope to build intuition around D_2^α that facilitates its application. I'll then refocus on the discursive dilemma, showing the advantages of D_2^α over D_2. I conclude by mentioning similarities with the modal probability logic advanced by Heifetz and Mongin [3], but leave an in-depth exploration of the connection between the two logics for further research.

2 Background

2.1 Some Motivation

Classical logic abstractly describes a set of valid inference patterns. Integral to its usual formulation, L_2, are two axiomatic assumptions that "non-classical" logics (usually) deviate from – the law of the excluded middle $\mathcal{P} \vee \neg\mathcal{P}$, and the law of non-contradiction $\neg(\mathcal{P} \wedge \neg\mathcal{P})$. For our purposes, we will look at the latter–the law of non-contradiction. This principle has roots in Aristotelian logic – "The principle that two contradictory statements are not both true is the most certain of all" (quoted after Jaskowski/Lukasiewicz) [5] [8]. For a single logician, this principle is sound – we certainly want to invalidate formulae of the form $(\mathcal{P} \wedge \neg\mathcal{P})$, and we certainly call anyone who holds contradictory beliefs illogical or irrational. This axiom along with the principle *ex falso quodlibet* ($\bot \models \mathcal{Q}$) gives rise to $(\mathcal{P} \wedge \neg\mathcal{P}) \models \mathcal{Q}$. From adjunction ($p, q \models p \wedge q$), we can then (classically) conclude $\mathcal{P}, \neg\mathcal{P} \models \mathcal{Q}$.

This L_2 phenomenon is often described as *overfilling* or *explosion*, because in the presence of contradictory premises, anything can be inferred. We formally define the *implicational law of overfilling* as $p \to (\neg p \to q)$. For a consistent system – one that does not affirm a proposition and its negation – the implicational law of overfilling (and its variants) is of no concern. Yet as soon as a system becomes *inconsistent* – one that affirms a proposition and its negation (i.e. contains a contradiction) – that system becomes *overfilled*.

What motivation is there for studying *inconsistent* systems? By their nature, they violate an ancient logical principle. Let's consider, however, the difficulties posed by judgment aggregation within the framework of L_2. Suppose we have a group of $\mathcal{D}$

[3] I sometimes use the variant $\diamond_\alpha$ where α refers to some fraction i/d for brevity

discussants, trying to reach a consensus on a set of $\mathcal{P}$ propositions. For each proposition p, each has the opportunity to either agree that p or deny that p (here p can be any wff of L_2). Our goal is to aggregate those decisions into a non-contradictory group consensus. That is, we want to develop some way to aggregate $|\mathcal{D}|$ individual judgments on $|\mathcal{P}|$ propositions into a collective, *consistent*, agreement. A reasonable constraint in this set-up is that for *every* proposition, $p_i \in \mathcal{P}$, *every* discussant, $d_i \in \mathcal{D}$, will either hold that p_i or that $\neg p_i$. Another reasonable assumption is that for any contradictory $\Gamma, \Psi \in \mathcal{P}$, no discussant d_i holds both Γ, Ψ. More concisely, every discussant is a classically consistent logician that makes an assertion about every proposition.

In this set-up, there are $\binom{2^{|\mathcal{P}|}+|\mathcal{D}|-1}{2^{|\mathcal{P}|}-1}$ possible configurations of the discussants. Naturally, in most of these configurations there will be disagreement among the discussants. Some may advance p while others advance $\neg p$ – that is the nature of rational discourse. Yet, in these situations L_2 is insufficient. Without imposing some *aggregation rule*, any contradictory opinions among discussants, (d_i holds Γ true, while d_j holds Γ false) creates the adjunction $\Gamma, \neg\Gamma$ in aggregation, and thus overfills the entire system. Enforcing a *majority rule* constraint – i.e. p is true if and only if more than half of the discussants hold that p – solves this problem (as it will never be the case that more than half holds that p *and* more than half holds that $\neg p$). But as will be shown, in the presence of a *majority rule*, our L_2 connectives are no longer reliable. That is, L_2 operators behave unexpectedly when combined with the modal operator *the majority holds that*, used here implicitly.

In the following sections, I will describe (and subsequently extend) a system that models rational discourse in light of these limitations, Stanislaw Jaskowski's *discussive* logic, D_2. Before, describing D_2 at length however, a more formal discussion of Condorcet's paradox and the so-called discursive dilemma is necessary.

2.2 Condorcet's Paradox and the Discursive Dilemma

Consider the following situation: there are 3 choices, A, B, C which 3 voters, d_1, d_2, d_3, strictly rank. These voters cast the following ballots:

d_1	$A > B > C$
d_2	$C > A > B$
d_3	$B > C > A$

Table 1: A cyclic set of ballots representing Condorcet's Paradox

Using the majority rule, we have that 2 out of 3 voters hold $A > B$, and further that 2 out of 3 hold $B > C$. However, we also have that 2 out of 3 hold $C > A$. Aggregating these gives the result $A > B > C > A$, clearly an impossible situation. This is Condorcet's Paradox – it arises whenever a "voting cycle" exists in a ranked ballot system. If we introduce variables for each pairwise relationship, the connection with the principle of non-contradiction becomes more clear. That is, let α represent $A > B$, $\beta : B > C$, $\gamma : C > A$

Rewriting the previous table using these new variables we have:

d_1	α	β	$\neg\gamma$
d_2	α	$\neg\beta$	γ
d_3	$\neg\alpha$	β	γ

Table 2: The same cyclic set of ballots represented symbolically

Thus, the aggregation of ballots produces pairs $(\alpha, \neg\alpha), (\beta, \neg\beta)$, and $(\gamma, \neg\gamma)$. By the previous discussion, these pairs imply that without a majority rule, L_2 is insufficient for making inferences about group consensus. However, *with* a majority rule we have α, β, and γ, the contradictory case. Notice though, that there is *not* majority support for any of $(\alpha \wedge \beta), (\alpha \wedge \gamma)$, or $(\beta \wedge \gamma)$. This feature of Condorcet's Paradox intimates the more general impossibility result mentioned earlier and studied extensively by List and Pettit, the *discursive dilemma*. A simple formulation of this dilemma (sometimes referred to as the "doctrinal paradox") simplifies the discussion to only consider two propositions α and β and the result of their conjunction, $\alpha \wedge \beta$. The following table demonstrates the paradox:

	α	β	$\alpha \wedge \beta$
d_1	True	True	True
d_2	True	False	False
d_3	False	True	False
$Majority$	True	True	False

Table 3: The discursive dilemma (using conjunction)

Just as in Condorcet's Paradox, we see that 2 out of 3 of the voters hold that α, 2 out of 3 hold that β, but only 1 out of 3 holds that $\alpha \wedge \beta$. Clearly then, the truth-values of α, β, and $\alpha \wedge \beta$ are not well defined by L_2. More precisely speaking, we see that if we define $\gamma \leftrightarrow \alpha \wedge \beta$, then the majority opinion given by $\{\alpha, \beta, \neg\gamma\}$ is *inconsistent*.

This problem proves to be quite general [6] [7]. Inconsistent aggregation results are not limited to conjunctions of propositions. The following table demonstrates the discursive dilemma using implication:

	α	β	$\alpha \to \beta$
d_1	True	True	True
d_2	True	False	False
d_3	False	False	True
$Majority$	True	False	True

Table 4: The discursive dilemma (using implication)

This substantiates the claim that, when applying a *majority rule* to a system of rational agents, logical connectives behave unexpectedly. Further, because the aggregation results may be *inconsistent*, any such system may also be overfilled. This discussion strongly suggests that L_2 is simply incapable of modeling discourse meaningfully. Thus the goal is two-fold – to provide a logic that doesn't overfill a system that contains inconsistent premises and to offer connectives in that logic that behave analogously to those in L_2. In the next section, I introduce D_2, a logic with that specific goal in mind.

2.3 D_2 and the Modal Logic S_5

The Polish logician Stanislaw Jaskowski, largely motivated by concerns regarding the aforementioned implicational law of overfilling, sought to model multiple rational agent systems. In *A Propositional Calculus for Inconsistent Deductive Systems*, he advances D_2, a two-valued *discussive* logic. Importantly, D_2 is a "non-adjunctive paraconsistent logic". That is, it does not validate adjunctive *ex falso quodlibet* ($\mathcal{P}, \neg\mathcal{P} \models \mathcal{Q}$). Therefore, as the title of Jaskowski's paper implies, D_2 facilitates the analysis of *inconsistent* deductive systems. We have already shown that the previous set up of $\mathcal{P}$ and $\mathcal{D}$ represents such a system because while no discussant d_i will advance both p and $\neg p$[4], it may be (and is in fact likely to be) the case that one discussant advances p while another advances $\neg p$.

Jaskowski begins his pursuit with the question, "Can a discussive logic be based on 2-valued logic?" [5]. Here a "discussive" logic can be thought of as a logical model for judgment aggregation. The previous section introduced the difficulty of the problem and considering the discursive dilemma using implication, we see that even the elementary form of reasoning *modus ponens* ($p, p \to q \models q$) fails. In a judgment aggregation setting, we may have both p, and $p \to q$ without having q (if p, $p \to q$ are held by different discussants). Jaskowksi's goal in creating D_2 then was to define "discussive" equivalents of $\to$ and $\leftrightarrow$ that retain the useful and expected behavior of their non-discussive analogs.

As we will see, D_2 is closely related to the modal logic M_2 and the unary operators $\diamond$, $\Box$ – or *it is possible that*, and *it is necessary that* respectively. Jaskowski defines two connectives, $\to_d, \leftrightarrow_d$ for *discussive implication* and *discussive equivalence* as follows:

$$p \to_d q \equiv \diamond p \to q \tag{2}$$

$$p \leftrightarrow_d q \equiv (\diamond p \to q) \wedge (\diamond q \to \diamond p) \tag{3}$$

With these connectives in place, he defines the theses of D_2 as:

> The set of formula constructed using only the connectives $\neg, \wedge, \vee, \to_d$, and $\leftrightarrow_d$ such that when preceded by $\diamond$ they become M_2 theses. [5]

In a trivial example,

> $D_2 1$
> $p \to_d p \quad \equiv \quad \diamond p \to p$
> (if it is *possible* that p then p)

We see that though it is a D_2 thesis, it only becomes an M_2 thesis when preceded with a $\diamond$. That is,

[4] by the consistency constraint

$M_2 1$

$\diamond(\diamond p \rightarrow p)$

To understand why we hold $M_2 1$, we must understand how to simplify expressions with repeated modal operators. For this, we need to first decide which axiomatization of M_2 we will employ. We choose the S_5 axiomatization of M_2 which stipulates that:

$$XX...X\Box p \equiv \Box p \tag{4}$$

$$XX...X \diamond p \equiv \diamond p \tag{5}$$

Where each X is either a $\Box$ or $\diamond$. With these axioms defined, we now just need laws for distribution of modal operators over implication, equivalence, and disjunction. Jaskowski defines the following 3 M_2 theorems for this purpose:

$$\diamond(p \rightarrow_d q) \leftrightarrow (\diamond p \rightarrow \diamond q) \tag{6}$$

$$\diamond(p \leftrightarrow_d q) \leftrightarrow (\diamond p \leftrightarrow \diamond q) \tag{7}$$

$$\diamond(p \vee q) \leftrightarrow (\diamond p \vee \diamond q) \tag{8}$$

With the S_5 axiomatization in mind, and these definitions provided, it's easy to see that $\diamond(\diamond p \rightarrow p)$ reduces to $\diamond p \rightarrow \diamond p$. Which is clearly an M_2 thesis.

Using these new connectives we can show that *modus ponens* $(\mathcal{P}, \mathcal{P} \rightarrow_d \mathcal{Q} \models \mathcal{Q})$ may be preserved if discussive implication is used in place of ordinary implication. As Jaskowski explains, this result depends on the M_2 theorem $\diamond(\diamond p \rightarrow q) \rightarrow (\diamond p \rightarrow \diamond q)$ which follows from (5) and (6). The D_2 maintenance of *modus ponens* suggests that it may be more adept at modeling the discursive dilemma. The following section investigates that claim.

2.4 Applying D_2 to the Discursive Dilemma

Referring to Table 3, we see that $\diamond\alpha$, $\diamond\beta$ and $\diamond(\alpha \wedge \beta)$ are all true (and therefore D_2 validates each). Thus, the new "aggregation rule" that D_2 employs doesn't produce the same inconsistent results when applied to the discursive dilemma. Referring to Table 4, we see that α discussively implies β, similarly preventing the inconsistency if we use $\rightarrow_d$ in place of $\rightarrow$. This feature, along with its rejection of the implicational law of overfilling $(p \rightarrow_d (\neg p \rightarrow_d q))$, suggests that D_2 handles the paradox more meaningfully. However, there is a certain sense in which information from the ballots is lost by applying this rule.

From a purely logical perspective, D_2 neatly resembles classical logic and facilitates inference in a judgment aggregation setting. But to see the practical limitations, consider the following two situations. In each, only a single discussant holds both α and β:

	α	β	$\alpha \wedge \beta$
d_1	False	False	False
d_2	False	False	False
d_3	False	False	False
$\vdots$	$\vdots$	$\vdots$	$\vdots$
d_{99}	False	False	False
d_{100}	False	False	False
d_{101}	True	True	True

Table 5: A set of ballots where a single discussant validates both α and β

	α	β	$\alpha \wedge \beta$
d_1	True	False	False
d_2	True	False	False
d_3	True	False	False
$\vdots$	$\vdots$	$\vdots$	$\vdots$
d_{49}	True	False	False
d_{50}	True	False	False
d_{51}	False	True	False
$\vdots$	$\vdots$	$\vdots$	$\vdots$
d_{99}	False	True	False
d_{100}	False	True	False
d_{101}	True	True	True

Table 6: A different set of ballots where a single discussant validates both α and β

In the first set of ballots, we see that an overwhelming majority negate both α and β, and a single discussant affirms both. The second set is an expanded version of the discursive dilemma. Again, only a single discussant affirms both propositions, but the remaining discussants are evenly split between the two. Intuitively, these situations differ immensely. In the first, we can easily conclude that the majority holds *neither* α nor β despite the lone dissenter. In the second, the disagreement is more prolific – there are two opposing factions and a lone compromiser.

Using D_2, however, this intuitive understanding is completely lost. In both situations, α, β and $\alpha \wedge \beta$ are *possible* (and therefore validated by D_2). While this leads to *consistent* aggregations, it does so at the expense of useful information – some of which can even be intuitively gleaned from the ballots. Thus, discussive implication and equivalence are fairly weak notions of their L_2 analogs – only that of possibility. If we instead consider the stronger notion of *probability*, we can obtain systems that more precisely capture the nature of democratic disagreement. This hints at the approach of strengthening the $\diamond$ operator. If we consider the modal operator defined to be *i out of d discussants hold that* or *it is i/d probable that*, we may begin to unpack problem. The remainder of this paper will focus on extending D_2 using the unary modal operator $\diamond_{i/d}$[5] defined by (1) in the introduction.

[5]Sometimes I will use the notation $\diamond_\alpha$ where $\alpha \in [1/d, 1]$ for brevity

3 D_2^α and the Democratic Logics

3.1 Defining democratic implication and equivalence, $\to_{i/d}$ and $\leftrightarrow_{i/d}$

The modal operator $\diamond_{i/d}$ is incredibly flexible – it encapsulates notions of possibility, necessity, and any fractional values in between for various choices of i. Following the example set by D_2 we may define connectives for *democratic equivalence* and *democratic implication*, $\leftrightarrow_{j/d}, \to_{j/d}$[6] as follows:

$$p \to_{j/d} q \equiv \diamond_{j/d} p \to q \tag{9}$$

$$p \leftrightarrow_{j/d} q \equiv (\diamond_{j/d} p \to q) \wedge (\diamond_{j/d} q \to \diamond_{j/d} p) \tag{10}$$

To facilitate using these connectives, we update (4) and (5) for our new operator:

$$XX...X \diamond_{i/d} p \equiv \diamond_{i/d} p \tag{11}$$

where each X represents $\diamond_{i/d} \forall i : 1 \le i \le d$

With these connectives' behaviors well defined – we can generate logics similar to D_2 for specific choices of i. Before proceeding with that discussion, a notational clarification is useful. In this section, I referred to the modal probability operator using fractions i/d, but to the implicational/equivalence operators using fractions j/d. This choice was deliberate and is intimately related to the definition of a specific logic $D_2^{(i/d)}$. For now, simply think of i as a reserved variable – its notational importance will be elucidated in the following section.

3.2 Defining the set D_2^α

We can now form a family of *democratic* logics, defined by nearly the same conditions as Jaskowski's D_2. Each logic in this family of logics consists of a set of formulae $\mathcal{T}$, theses of $D_2^{(i/d)}$ (for a specific choice of i), such that for each, the following holds:

1. $\mathcal{T}$ includes sentential variables and at most the following operators: $\neg, \wedge, V, \to, \leftrightarrow, \to_{j/d}$, and $\leftrightarrow_{j/d}$ $\forall j \le d$ [7]
2. Preceding $\mathcal{T}$ with the operator $\diamond_{i/d}$ yields a theorem in the two-valued logic M_2 (using the axiomatization of S_5)

The importance of i can now be stated more clearly. While the choice of i defines a specific logic $D_2^{(i/d)}$, we may use connectives $\to_{j/d}$ and $\leftrightarrow_{j/d}$ for each $j \le d$ (i.e. even for $j \ne i$). This facilitates the desired flexibility for describing judgment aggregation problems, preserving the maximum amount of information from an inconsistent deductive system. For clarity then, I will *always* use i to refer to the choice of $\diamond_{i/d}$ that may

[6]Sometimes $\leftrightarrow_\beta, \to_\beta$

[7]Again, $\diamond_\alpha$, and $\to_\beta$, $\leftrightarrow_\beta$ are often neater without being ambigious. When the meaning is ambiguous, I use the verbose notation. Otherwise, I use the abbreviated form.

precede any $D_2^{(i/d)}$ thesis to produce an M_2 thesis. Any other variables (j, k, l, etc.) will refer to dummy variables, specific to the implication or equivalence relation they follow.

As the following analyses will show, with specific restrictions on the types of democratic implication and equivalence employed, various L_2 theses are either retained or invalidated. With access to a multitude of connectives, an informed choice engenders a much deeper sense of interpretability when modeling an inconsistent system.

To develop an intuition around the family of democratic logics described, the following section compares some D_2 theorems with their counterparts in $D_2^{(i/d)}$, and provide what restrictions (given the choice of i) are necessary. Before proceeding, it's useful to consider the following theorems which help to reduce formulae:

$$\diamond_j p \rightarrow \diamond_k p \tag{12}$$

True $\forall k, j : 1 \leq k \leq j \leq d$

$$\neg \diamond_{h/d} p \leftrightarrow \diamond_{j/d} \neg p \tag{13}$$

True $\forall h, j : h + j > d$

The first is fairly straight forward and says that when at least j discussants hold p, then the same is true for every k less than j. The second is the law of distribution for negation over $\diamond_{i/d}$. The M_2 theorem $\neg\Box p \leftrightarrow \diamond\neg p$, is a specific instance of this theorem for $j = 1$ and $h = d$. With these in mind, we can now begin the project of extending D_2 into $D_2^{(i/d)}$

3.3 D_2 Methodological Theorems revisited

In the following sections, we will analyze a number of D_2 theses, presented originally by Jaskowski. To achieve this, understanding the three methodological theorems he offers, and their applicability in this new system is useful.

> *D_2 Methodological Theorem 1:* Every thesis $\mathcal{T}$ in L_2 that only contains the connectives $\rightarrow, \leftrightarrow, \vee$ becomes a thesis $\mathcal{T}_d$ in D_2 when the implication symbols in $\mathcal{T}$ are replaced by $\leftrightarrow_d$, $\rightarrow_d$.

Jaskowski leverages this theorem tremendously throughout his paper. However, his proof relies on the fact that for each variable in $\mathcal{T}$, a substitution of the form $p/\diamond p$ occurs. In our system, unfortunately, in a single $D_2^{(i/d)}$ formula there may be many different operators $\rightarrow_{j/d}$, $\rightarrow_{k/d}$...etc. Thus, each switch $p/\diamond_\alpha p$ is not necessarily guaranteed. However, the theorem holds if we stipulate that each instance of $\rightarrow$ *and* $\leftrightarrow$ is replaced with $\rightarrow_{i/d}$ and $\leftrightarrow_{i/d}$ for the specific i. By doing this we ensure that each formula has only one modal operator, $\diamond_{i/d}$, thus the substitution $p/\diamond_{i/d} p$ is valid. With this condition satisfied, the proof follows directly from Jaskowski's paper.

D_2 *Methodological Theorem 2:* If $\mathcal{T}$ is a thesis in L_2 and includes at most the connectives, $\wedge$, $\vee$, and $\neg$. Then:

1. $\mathcal{T}$
2. $\neg\mathcal{T} \to \mathcal{Q}$

are theses of D_2.

This theorem holds in D_2 and in each $D_2^{(i/d)}$.

D_2 *Methodological Theorem 3:* If in a thesis T belonging to D_2, you replace each instance of $\to_d$ with $\to$ and each instance of $\leftrightarrow_d$ with $\leftrightarrow$, an L_2 thesis is obtained.

If we update this rule to say that we replace each instance of $\to_{i/d}$ and $\leftrightarrow_{i/d}$ in a $D_2^{(i/d)}$ thesis with $\to$ and $\leftrightarrow$, and further that the thesis contains no democratic implication or equivalence relations $\to_{j/d}, \leftrightarrow_{j/d}$ such that $j \neq i$, then it still holds.

Proof: Any M_2 thesis becomes an L_2 thesis once each modal operator is removed. By definition, $D_2^{(i/d)}$ theses when preceded with a $\diamond_i$ are M_2 theses. Expanding any instances of $\to_i$ or $\leftrightarrow_i$ only replaces them with $\to$ and $\leftrightarrow$ and the modal operator $\diamond_i$. Removing these yields an L_2 thesis. □

With these methodological theorems at our disposal, we can begin investigating D_2 theses under this new system.

4 Practical Applications of $D_2^{(i/d)}$

The following sections are included to supplement intuition about $D_2^{(i/d)}$ and to examine some of its practical advantages. In the first two sections, I examine a selection of D_2 theses and non-theses offered by Jaskowski. Where possible, I give restrictions on the choices of connectives (relative to each other and the choice of i) that constrain these formulae to be $D_2^{(i/d)}$ theses. In the final section, I readdress the discursive dilemma and the examples offered in tables 5 and 6 to show that $D_2^{(i/d)}$ is capable of producing consistent results without sacrificing useful information about the inconsistent system of interest.

4.1 D_2 Theses in $D_2^{(i/d)}$

$D_2 1$

$$p \to_{j/d} p \tag{14}$$

Holds $\forall i, j : 1 \leq i \leq j \leq d$

Interestingly, even in this trivial case, the restriction on j is necessary. Though the result follows from (12), it's helpful to examine it explicitly. Consider the system of 3 discussants in logic $D_2^{(2/3)}$ where $j = 1$. Here the formula reduces to $\diamond_{1/3} p \to \diamond_{2/3} p$. Clearly, we may have that $\diamond_{1/3} p$ without having $\diamond_{2/3} p$, invalidating the formula

With this result in mind, we can easily extend *modus ponens* for $D_2^{(i/d)}$. For clarity, I use the more modern presentation (rather than Jaskowski's esoteric formulation).

Modus Ponens

$$\mathcal{P}, \mathcal{P} \rightarrow_{j/d} \mathcal{Q} \models \mathcal{Q} \tag{15}$$

Holds $\forall i, j : 1 \leq j \leq i \leq d$

The first premise says that i discussants hold $\mathcal{P}$. The second says that if j discussants hold $\mathcal{P}$ then i discussants hold $\mathcal{Q}$. The conclusion says that i discussants hold $\mathcal{Q}$. Therefore by (14) the result follows.

The following can be shown by calculations – some examples are expanded for added clarity.

$D_2 2$

$$(p \leftrightarrow_{j/d} q) \leftrightarrow_{k/d} (q \leftrightarrow_{l/d} p) \tag{16}$$

Holds $\forall i, j, k, l : 1 \leq k \leq j \leq d$, and $1 \leq i \leq l \leq d$

$D_2 4$ (Law of contradiction)

$$\neg(p \wedge \neg p) \tag{17}$$

Holds by Methodological Theorem 2

$D_2 5$ (Conjunctional Law of overfilling)

$$(p \wedge \neg p) \rightarrow q \tag{18}$$

Holds by Methodological Theorem 2

$D_2 6$

$$(p \wedge q) \rightarrow_{j/d} p \tag{19}$$

Holds $\forall i, j : 1 \leq i \leq j \leq d$

$D_2 7$

$$p \rightarrow_{j/d} (p \wedge p) \tag{20}$$

Holds $\forall i, j : 1 \leq i \leq j \leq d$

$D_2 8$

$$(q \wedge p) \rightarrow_{j/d} (p \wedge q) \tag{21}$$

Holds $\forall i, j : 1 \leq i \leq j \leq d$

$D_2 9$

$$(p \wedge (q \wedge r)) \leftrightarrow_{j/d} ((p \wedge q) \wedge r) \tag{22}$$

Holds $\forall i, j : 1 \leq i \leq j \leq d$

$D_2 10$ (Law of importation)

$$(p \rightarrow_{j/d} (q \rightarrow_{k/d} r)) \rightarrow_{h/d} ((p \wedge q) \rightarrow_{l/d} r) \tag{23}$$

Holds $\forall i, h : 1 \leq i \leq h \leq d$,
$\forall j, k, l$: when $j + k \leq d$ then $min(j, k) \leq l \leq d$ or if $j + k > d$, then $l > (j + k) - d$

That is, the formula fails when:
Case 1: $i > h$

We may have $\diamond_{h/d} r$ but not $\diamond_{i/d} r$. So assume $/p/ = /q/ = 1$. By expanding each implication we have: $(\diamond_{j/d} p \rightarrow (\diamond_{k/d} q \rightarrow \diamond_{h/d} r)) \rightarrow (\diamond_{l/d}(p \wedge q) \rightarrow \diamond_{i/d} r)$

Clearly if p, q are tautologies and $\diamond_{h/d} r$ is true while $\diamond_{i/d} r$ is false, the formula is false.

Case 2: $l \leq j + k - d$ *(for* $j + k > d$*)*

Consider $d = 5, k = 4, j = 3, l = 1$. Here $j + k - d = 2$ larger than $l = 1$
From above we have:
$(\diamond_{3/5} p \rightarrow (\diamond_{4/5} q \rightarrow \diamond_{h/5} r)) \rightarrow (\diamond_{1/5}(p \wedge q) \rightarrow \diamond_{i/5} r)$
Clearly, when r is a contradiction and we have a small minority that holds $p \wedge q$ (i.e. $\diamond_{1/5}(p \wedge q)$ is true, but $\diamond_{3/5} p, \diamond_{4/5} q$ is false), then the formula is invalidated.

Case 3: $l \leq min(j, k)(j + k \leq d)$

Consider $d = 5, k = j = 2, l = 1$ From above we have:

$(\diamond_{2/5} p \rightarrow (\diamond_{2/5} q \rightarrow \diamond_{h/5} r)) \rightarrow (\diamond_{1/5}(p \wedge q) \rightarrow \diamond_{i/5} r)$

As in *Case 2*, when r is a contradiction we may have a small minority such that $\diamond_{2/5} p, \diamond_{2/5} q$ are false, but $\diamond_{1/5}(p \wedge q)$ is true, invalidating the formula

$D_2 13$

$$p \leftrightarrow_{j/d} \neg\neg p \tag{24}$$

Holds $\forall i, j : 1 \leq i \leq j \leq d$

$D_2 14$

$$(\neg p \rightarrow_{j/d} p) \rightarrow_{k/d} p \tag{25}$$

Holds $\forall i, j, k : 1 \leq i \leq k \leq d$, and $(i + j) \leq d + 1$

Case 1: $k < i$

Assume $k = 1, i = 2, d = 3$.
Clearly we may have $\diamond_{1/3} p$, $\diamond_{2/3} \neg p$, but not $\diamond_{2/3} p$.
Thus the formula is invalid.

Case 2: $i + j > d + 1$

Consider $d = 5, j = 4, i = 3$. Here $i + j = 7 > 5 + 1$.
If only 2 discussants hold that p. Then $\diamond_{4/5}\neg p$ false, and therefore $(\neg p \rightarrow_{4/5} p)$ is true.
But we do not have that $\diamond_{3/5}p$, only $\diamond_{2/5}p$
Thus the formula is invalid.

$D_2 15$

$$(p \rightarrow_{j/d} \neg p) \rightarrow_{k/d} \neg p \tag{26}$$

Holds $\forall i, j, k : 1 \leq i \leq k \leq d$, and $(i + j) \leq d + 1$

$D_2 16$

$$(p \leftrightarrow_{j/d} \neg p) \rightarrow_{k/d} p \tag{27}$$

Holds $\forall j, k : 1 \leq j \leq d$, and $1 \leq k \leq d$

$D_2 17$

$$(p \leftrightarrow_{j/d} \neg p) \rightarrow_{k/d} \neg p \tag{28}$$

Holds $\forall j, k : 1 \leq j \leq d$, and $1 \leq k \leq d$

$D_2 21$

$$(p \rightarrow_{j/d} (q \wedge \neg q)) \rightarrow_{k/d} \neg p \tag{29}$$

Holds $\forall i, j : (i + j) \leq d + 1$

$D_2 22$

$$(\neg p \rightarrow_{j/d} (q \wedge \neg q)) \rightarrow_{k/d} p \tag{30}$$

Holds $\forall i, j : (i + j) \leq d + 1$

$D_2 23$

$$\neg(p \leftrightarrow_{j/d} \neg p) \tag{31}$$

Holds $\forall j : 1 \leq j \leq d$

$D_2 24$

$$\neg(p \rightarrow_{j/d} q) \rightarrow_{k/d} p \tag{32}$$

Holds $\forall i, j, k : 1 \leq i \leq j \leq d$ and $1 \leq k \leq d$

$D_2 25$

$$\neg(p \rightarrow_{j/d} q) \rightarrow_{k/d} \neg q \tag{33}$$

Holds $\forall i, j, k : 1 \leq i \leq k \leq d$ and $1 \leq j \leq d$

$D_2 26$

$$p \rightarrow_{j/d} (\neg q \rightarrow_{k/d} \neg(p \rightarrow_{l/d} q)) \tag{34}$$

Holds $\forall i, k, l : 1 \leq i \leq k \leq d$, and $1 \leq l \leq i \leq d$

Case 1: $k < i$

$D_2 26$ reduces to:
$\diamond_{j/d} p \to (\diamond_{k/d} \neg q \to \diamond_{i/d} \neg (\diamond_{l/d} p \to q))$
Choosing $k = 1, i = 2, d = 3$, produces:
$\diamond_{j/d} p \to (\diamond_{1/3} \neg q \to \diamond_{2/3} \neg (\diamond_{l/d} p \to q))$
We see that when $\diamond_{1/3} \neg q$ is true, q is either false or held false by a single discussant (i.e. $\diamond_{2/3} \neg q$ is false) When p is a tautology, and $\diamond_{1/3} \neg q$ is true, we then have:
$\diamond_{2/3} \neg (\diamond_{l/d} p \to q)$ false, but
$\diamond_{1/3} \neg (\diamond_{l/d} p \to q)$ true
Invalidating the formula.

Case 2: $i > l$

Assume $l = 1, i = 2, d = 3$ From above we have: $\diamond_{j/d} p \to (\diamond_{k/d} \neg q \to \diamond_{2/3} \neg (\diamond_{1/3} p \to q))$
Consider the case when a lone discussant holds that p, and all hold q false. Then we have $\diamond_{1/3} p$ true, but not $\diamond_{2/3} p$. Therefore:
$\diamond_{2/3} \neg (\diamond_{1/3} p \to q)$ is false, invalidating the formula.
Intuition for this problem comes from equation (13)

4.2 Non-D_2 Theses with the new implication rule

$(nonD_2)1$

$$p \to_{j/d} (q \to_{k/d} (p \wedge q)) \tag{35}$$

Holds $\forall i, j, k : 1 \leq j + k - d \leq i \leq d$

$(nonD_2)2$ Law of exportation:

$$((p \wedge q) \to_{j/d} r) \to_{k/d} (p \to_{l/d} (q \to_{m/d} r)) \tag{36}$$

Fails for all assignments of i, j, k, l, m

$(nonD_2)3$ Implicational law of overfilling:

$$p \to_{j/d} (\neg p \to_{k/d} q) \tag{37}$$

Holds $\forall j, k : j + k > d$

$(nonD_2)3a$

$$p \to_{j/d} (\neg p \to_{k/d} \neg q) \tag{38}$$

Holds $\forall j, k : j + k > d$

$(nonD_2)6$

$$(p \to_{j/d} \neg p) \to_{k/d} ((\neg p \to_{l/d} p) \to_{h/d} q) \tag{39}$$

Holds $\forall j, k, l, h : j + k > d$, and $l + h > d$

$(nonD_2)6a$

$$(p \rightarrow_{j/d} \neg p) \rightarrow_{k/d} ((\neg p \rightarrow_{l/d} p) \rightarrow_{h/d} (p \wedge \neg p)) \tag{40}$$

Holds $\forall j, k, l, h : j + k > d$, and $l + h > d$

$(nonD_2)7$

$$\neg(p \rightarrow_{j/d} p) \rightarrow_{k/d} q \tag{41}$$

Fails $\forall j, k$

$(nonD_2)8$

$$(p \rightarrow_{j/d} q) \rightarrow_{k/d} (\neg q \rightarrow_{l/d} \neg p) \tag{42}$$

Holds $\forall i, j, k, l : j + i > d$, and $k + l > d$

$(nonD_2)9$

$$(\neg p \rightarrow_{j/d} \neg q) \rightarrow_{k/d} (q \rightarrow_{l/d} p) \tag{43}$$

Holds $\forall i, j, k, l : j + i > d$, and $k + l > d$

$(nonD_2)10$

$$(p \rightarrow_{j/d} q) \rightarrow_{k/d} ((p \rightarrow_{l/d} \neg q) \rightarrow_{h/d} \neg p) \tag{44}$$

Holds $\forall i, j, k, l, h : 1 \leq l \leq j \leq d$, and $j + i > d$, and $k + h > d$

$(nonD_2)11$

$$(\neg p \rightarrow_{j/d} q) \rightarrow_{k/d} ((\neg p \rightarrow_{l/d} \neg q) \rightarrow_{h/d} p) \tag{45}$$

Holds $\forall i, j, k, l, h : 1 \leq l \leq j \leq d$, and $j + i > d$, and $k + h > d$

Though some theses presented in Jaskowski's original paper have been omitted (as evidenced by the gaps in the numbering), the ones presented here adequately cover the topics expressed by those.

4.3 Applying $D_2^{(i/d)}$ in the previous scenarios

If we now reconsider the situations presented in tables 5 and 6 (used to describe the limitations of D_2) using a logic in D_2^{α}, we see that we can retain desirably consistent results without the same information loss. In each, $\alpha \wedge \beta$ only holds in $D_2^{(1/101)}$, but in the former both α and β also only hold in $D_2^{(1/101)}$. In the latter, however, α and β both hold in $D_2^{(51/101)}$. For these simple scenarios, a complex logical model seems unwieldy (and perhaps unnecessary), but in a more nuanced example the advantage of D_2^{α} is more immediate. Consider the following two sets of ballots, analogous to those presented in tables 5 and 6. In the first a simple majority rule correctly suggests that the voters hold α, but neither β nor $\alpha \rightarrow \beta$. The second is another expanded example of the discursive dilemma – this time using implication.

	α	β	$\alpha \to \beta$
d_1	True	False	False
d_2	True	False	False
$\vdots$	$\vdots$	$\vdots$	$\vdots$
d_{100}	True	False	False
d_{101}	True	True	True
$Majority$	True	False	False

Table 7: A set of ballots where α discussively implies β

	α	β	$\alpha \to \beta$
d_1	True	True	True
d_2	True	True	True
$\vdots$	$\vdots$	$\vdots$	$\vdots$
d_{50}	True	True	True
d_{51}	True	False	False
d_{51}	False	False	True
$\vdots$	$\vdots$	$\vdots$	$\vdots$
d_{100}	False	False	True
d_{101}	False	False	True
$Majority$	True	False	True

Table 8: An alternate set of ballots where α discussively implies β

In D_2, we have that in each set of ballots $\alpha \to_d \beta$. Just as before, the intuition provided by a simple majority rule fails to differentiate the two situations. If we consider these ballots using the more general system provided above, we can easily see that in the first $\alpha \to_{1/101} \beta$ is true for $D_2^{(1/101)}$ (this is equivalent to saying it holds in D_2). In the second, however, we can say that $\alpha \to_{51/101} \beta$ in $D_2^{(i/101)}, \forall i : 1 \leq i \leq 50$ – capturing substantially more information about the system in question. This is exactly the kind of flexibility that the set D_2^{α} is capable of providing.

5 Summary and Future Research

5.1 Modal Probability Logic

Before concluding, it's worth mentioning the similarities between the work presented here as D_2^{α} and the so-called modal logic of probability presented by Heifetz and Mongin in 1998 [3]. To facilitate the axiomatization of this logic, they introduce the belief operator L_α defined for rational $\alpha \in [0, 1]$, interpreted as "the probability is at least α". In this way, they extend modal logic to consider mappings from possible worlds to probability measures, rather than sets. Using L_α, they go on to define other similar operators M_α ($M_\alpha \psi \leftrightarrow L_{1-\alpha} \neg \psi$), E_α ($E_\alpha \psi \leftrightarrow L_\alpha \psi \wedge M_\alpha \psi$), interpreted as "the probability is at most α" and "the probability is exactly α" respectively. The modal operator I have introduced

$\diamond_{i/d}$ (used to define a logic $D_2^{(i/d)}$) captures the exact same notion as L_α, if we consider the *proportion* of discussants that hold p as the *probability* of p. Their model-theoretic approach differs strongly from mine, but suggests an avenue for further research. If (as I suspect) $\diamond_\alpha$ is semantically equivalent to L_α then a great deal more can easily be said about D_2^α based on their work. This exceeds the scope of this paper, but is an interesting topic for further research.

5.2 Final Remarks

As the previous sections hopefully have made evident, the family of logics, D_2^α, are particularly adept at modeling systems of rational discourse without losing information to over-simplification. I claim that this makes them useful for the practical analysis of judgment aggregation problems. Though they cannot resolve the well-studied impossibility results posed by such problems, an intelligent choice of i and use of connectives guarantee consistent aggregation results *without* reducing those results to incredibly weak statements. Thus, this extension of D_2 is pragmatically (rather than semantically) driven. That is, while D_2 disregards information in the pursuit of logically "neat" results, D_2^α maintains those same results without sacrificing potentially elucidatory information. Though the logic itself is somewhat "messier", the practical benefits are significant.

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