1 Introduction

This paper offers a paraconsistent logical framework for analyzing collective judgments. The framework presented is useful for clarifying the extent to which various propositions (and inferences based on those propositions) are proportionally true in group-discourse settings. Formally, this task is called judgment aggregation, which attempts to answer the following question (posed by Christian List):

“How can a group of individuals make consistent collective judgments on a set of propositions on the basis of the group members’ individual judgments on them?” [6]

Though the difficulty of judgment aggregation has been evident since the introduction of Condorcet’s paradox in the 18th century\(^1\), the field remained relatively inactive until the mid-20th century when Kenneth Arrow won the Nobel Prize for his seminal impossibility result. Since then (and particularly in the last 20 years), the field has exploded with research spanning multiple disciplines [6].

This paper will not consider Arrow’s Theorem explicitly, but will instead focus on a more general impossibility result—the so-called discursive dilemma or doctrinal paradox\(^2\), which presents the problem of judgment aggregation in a logical setting. This impossibility result was first formalized by List and Pettit in 2002, but has since been extended numerous times [6] [1] [2]. This paper will examine the applicability of Stanisław Jaskowski’s non-adjunctive two-valued logic, \(D_2\), to the discursive dilemma. As I will show, though \(D_2\) produces “consistent” aggregation results, its formulation loses or ignores practical information about the judgment aggregation problem under investigation. In light of this, I propose an extension of \(D_2\) that retains maximal inferential

\(^1\)First presented in 1785
\(^2\)For a detailed discussion of the relationship between Arrow’s Theorem and the discursive dilemma, see List and Pettit 2004 [7]
information in the judgment aggregation setting without producing “inconsistent” results. To achieve this, I introduce a new modal operator: $\diamond i/d$, defined as a unary operator such that:

$$\diamond i/d p \equiv p \text{ is held true in } i \text{ out of } d \text{ worlds}$$  \hspace{1cm} (1)

Using this operator, I define a family of democratic logics, nearly identical in structure to $D_2$, called $D_2^\alpha$. I say a family of logics because $\alpha$ represents a parameter whose value defines a specific logic. That is for a given choice of $\alpha$, say 2/3, each thesis in the logic $D_2^{(2/3)}$ becomes a thesis of the two-valued logic modal $M_2$ when preceded with the operator $\diamond 2/3$. Using Jaskowski’s 1948 paper [4] [5] as a guide, I will examine a number of $D_2$ theses and non-theses, comparing them with various $D_2^\alpha$ counterparts. Through this discussion, I hope to build intuition around $D_2^\alpha$ that facilitates its application. I’ll then refocus on the discursive dilemma, showing the advantages of $D_2^\alpha$ over $D_2$. I conclude by mentioning similarities with the modal probability logic advanced by Heifetz and Mongin [3], but leave an in-depth exploration of the connection between the two logics for further research.

2 Background

2.1 Some Motivation

Classical logic abstractly describes a set of valid inference patterns. Integral to its usual formulation, $L_2$, are two axiomatic assumptions that “non-classical” logics (usually) deviate from – the law of the excluded middle $P \lor \neg P$, and the law of non-contradiction $\neg (P \land \neg P)$. For our purposes, we will look at the latter—the law of non-contradiction. This principle has roots in Aristotelian logic – “The principle that two contradictory statements are not both true is the most certain of all” (quoted after Jaskowski/Lukasiewicz) [5] [8]. For a single logician, this principle is sound – we certainly want to invalidate formulae of the form $(P \land \neg P)$, and we certainly call anyone who holds contradictory beliefs illogical or irrational. This axiom along with the principle ex falso quodlibet ($\bot \models \bot$) gives rise to $(P \land \neg P) \models \bot$. From adjunction $(p, q \models p \land q)$, we can then (classically) conclude $P, \neg P \models \bot$.

This $L_2$ phenomenon is often described as overfilling or explosion, because in the presence of contradictory premises, anything can be inferred. We formally define the implicational law of overfilling as $p \rightarrow (\neg p \rightarrow q)$. For a consistent system – one that does not affirm a proposition and its negation – the implicational law of overfilling (and its variants) is of no concern. Yet as soon as a system becomes inconsistent – one that affirms a proposition and its negation (i.e. contains a contradiction) – that system becomes overfilled.

What motivation is there for studying inconsistent systems? By their nature, they violate an ancient logical principle. Let’s consider, however, the difficulties posed by judgment aggregation within the framework of $L_2$. Suppose we have a group of $D$
discussants, trying to reach a consensus on a set of $\mathcal{P}$ propositions. For each proposition $p$, each has the opportunity to either agree that $p$ or deny that $p$ (here $p$ can be any wff of $L_2$). Our goal is to aggregate those decisions into a non-contradictory group consensus. That is, we want to develop some way to aggregate $|\mathcal{D}|$ individual judgments on $|\mathcal{P}|$ propositions into a collective, consistent, agreement. A reasonable constraint in this setup is that for every proposition, $p_i \in \mathcal{P}$, every discussant, $d_i \in \mathcal{D}$, will either hold that $p_i$ or that $\neg p_i$. Another reasonable assumption is that for any contradictory $\Gamma, \Psi \in \mathcal{P}$, no discussant $d_i$ holds both $\Gamma, \Psi$. More concisely, every discussant is a classically consistent logician that makes an assertion about every proposition.

In this set-up, there are $\binom{2^{|\mathcal{P}|} + |\mathcal{D}| - 1}{2^{|\mathcal{P}|} - 1}$ possible configurations of the discussants. Naturally, in most of these configurations there will be disagreement among the discussants. Some may advance $p$ while others advance $\neg p$ – that is the nature of rational discourse. Yet, in these situations $L_2$ is insufficient. Without imposing some aggregation rule, any contradictory opinions among discussants, $(d_i$ holds $\Gamma$ true, while $d_j$ holds $\Gamma$ false) creates the adjunction $\Gamma, \neg \Gamma$ in aggregation, and thus overfills the entire system. Enforcing a majority rule constraint – i.e. $p$ is true if and only if more than half of the discussants hold that $p$ – solves this problem (as it will never be the case that more than half holds that $p$ and more than half holds that $\neg p$). But as will be shown, in the presence of a majority rule, our $L_2$ connectives are no longer reliable. That is, $L_2$ operators behave unexpectedly when combined with the modal operator the majority holds that, used here implicitly.

In the following sections, I will describe (and subsequently extend) a system that models rational discourse in light of these limitations, Stanislaw Jaskowski’s discussive logic, $D_2$. Before, describing $D_2$ at length however, a more formal discussion of Condorcet’s paradox and the so-called discursive dilemma is necessary.

### 2.2 Condorcet’s Paradox and the Discursive Dilemma

Consider the following situation: there are 3 choices, $A, B, C$ which 3 voters, $d_1, d_2, d_3$, strictly rank. These voters cast the following ballots:

<table>
<thead>
<tr>
<th></th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A&gt;B$</td>
<td>$B&gt;A$</td>
<td>$C&gt;A$</td>
<td></td>
</tr>
<tr>
<td>$B&gt;C$</td>
<td>$C&gt;B$</td>
<td>$B&gt;C$</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: A cyclic set of ballots representing Condorcet’s Paradox

Using the majority rule, we have that 2 out of 3 voters hold $A>B$, and further that 2 out of 3 hold $B>C$. However, we also have that 2 out of 3 hold $C>A$. Aggregating these gives the result $A>B>C>A$, clearly an impossible situation. This is Condorcet’s Paradox – it arises whenever a “voting cycle” exists in a ranked ballot system. If we introduce variables for each pairwise relationship, the connection with the principle of non-contradiction becomes more clear. That is, let $\alpha$ represent $A>B$, $\beta : B>C$, $\gamma : C>A$
Rewriting the previous table using these new variables we have:

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_1$</td>
<td>$\alpha$</td>
<td>$\beta$</td>
<td>$\neg \gamma$</td>
</tr>
<tr>
<td>$d_2$</td>
<td>$\alpha$</td>
<td>$\neg \beta$</td>
<td>$\gamma$</td>
</tr>
<tr>
<td>$d_3$</td>
<td>$\neg \alpha$</td>
<td>$\beta$</td>
<td>$\gamma$</td>
</tr>
</tbody>
</table>

Table 2: The same cyclic set of ballots represented symbolically

Thus, the aggregation of ballots produces pairs $(\alpha, \neg \alpha)$, $(\beta, \neg \beta)$, and $(\gamma, \neg \gamma)$. By the previous discussion, these pairs imply that without a majority rule, $L_2$ is insufficient for making inferences about group consensus. However, with a majority rule we have $\alpha$, $\beta$, and $\gamma$, the contradictory case. Notice though, that there is not majority support for any of $(\alpha \land \beta), (\alpha \land \gamma)$, or $(\beta \land \gamma)$. This feature of Condorcet’s Paradox intimates the more general impossibility result mentioned earlier and studied extensively by List and Pettit, the discursive dilemma. A simple formulation of this dilemma (sometimes referred to as the “doctrinal paradox”) simplifies the discussion to only consider two propositions $\alpha$ and $\beta$ and the result of their conjunction, $\alpha \land \beta$. The following table demonstrates the paradox:

<table>
<thead>
<tr>
<th></th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\alpha \land \beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_1$</td>
<td>True</td>
<td>True</td>
<td>True</td>
</tr>
<tr>
<td>$d_2$</td>
<td>True</td>
<td>False</td>
<td>False</td>
</tr>
<tr>
<td>$d_3$</td>
<td>False</td>
<td>True</td>
<td>False</td>
</tr>
<tr>
<td>Majority</td>
<td>True</td>
<td>True</td>
<td>False</td>
</tr>
</tbody>
</table>

Table 3: The discursive dilemma (using conjunction)

Just as in Condorcet’s Paradox, we see that 2 out of 3 of the voters hold that $\alpha$, 2 out of 3 hold that $\beta$, but only 1 out of 3 holds that $\alpha \land \beta$. Clearly then, the truth-values of $\alpha$, $\beta$, and $\alpha \land \beta$ are not well defined by $L_2$. More precisely speaking, we see that if we define $\gamma \leftrightarrow \alpha \land \beta$, then the majority opinion given by $\{\alpha, \beta, \neg \gamma\}$ is inconsistent.

This problem proves to be quite general [6] [7]. Inconsistent aggregation results are not limited to conjunctions of propositions. The following table demonstrates the discursive dilemma using implication:

<table>
<thead>
<tr>
<th></th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\alpha \rightarrow \beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_1$</td>
<td>True</td>
<td>True</td>
<td>True</td>
</tr>
<tr>
<td>$d_2$</td>
<td>True</td>
<td>False</td>
<td>False</td>
</tr>
<tr>
<td>$d_3$</td>
<td>False</td>
<td>False</td>
<td>True</td>
</tr>
<tr>
<td>Majority</td>
<td>True</td>
<td>False</td>
<td>True</td>
</tr>
</tbody>
</table>

Table 4: The discursive dilemma (using implication)

This substantiates the claim that, when applying a majority rule to a system of rational agents, logical connectives behave unexpectedly. Further, because the aggregation results may be inconsistent, any such system may also be overfilled. This discussion strongly suggests that $L_2$ is simply incapable of modeling discourse meaningfully. Thus
the goal is two-fold – to provide a logic that doesn’t overfill a system that contains inconsistent premises and to offer connectives in that logic that behave analogously to those in $L_2$. In the next section, I introduce $D_2$, a logic with that specific goal in mind.

2.3 $D_2$ and the Modal Logic $S_5$

The Polish logician Stanisław Jaskowski, largely motivated by concerns regarding the aforementioned implicational law of overfilling, sought to model multiple rational agent systems. In *A Propositional Calculus for Inconsistent Deductive Systems*, he advances $D_2$, a two-valued discursive logic. Importantly, $D_2$ is a “non-adjunctive paraconsistent logic”. That is, it does not validate adjunctive *ex falso quodlibet* ($\mathcal{P}, \neg \mathcal{P} \models \mathcal{Q}$). Therefore, as the title of Jaskowski’s paper implies, $D_2$ facilitates the analysis of inconsistent deductive systems. We have already shown that the previous set up of $\mathcal{P}$ and $\mathcal{D}$ represents such a system because while no discussant $d_i$ will advance both $p$ and $\neg p$, it may be (and in fact likely to be) the case that one discussant advances $p$ while another advances $\neg p$.

Jaskowski begins his pursuit with the question, “Can a discussive logic be based on 2-valued logic?” [5]. Here a “discussive” logic can be thought of as a logical model for judgment aggregation. The previous section introduced the difficulty of the problem and considering the discursive dilemma using implication, we see that even the elementary form of reasoning *modus ponens* ($p, p \rightarrow q \models q$) fails. In a judgment aggregation setting, we may have both $p$, and $p \rightarrow q$ without having $q$ (if $p$, $p \rightarrow q$ are held by different discussants). Jaskowski’s goal in creating $D_2$ then was to define “discussive” equivalents of $\rightarrow$ and $\leftrightarrow$ that retain the useful and expected behavior of their non-discussive analogs.

As we will see, $D_2$ is closely related to the modal logic $M_2$ and the unary operators $\Diamond, \Box$ – or *it is possible that*, and *it is necessary that* respectively. Jaskowski defines two connectives, $\rightarrow_d, \leftrightarrow_d$ for discursive implication and discursive equivalence as follows:

$$p \rightarrow_d q \equiv \Diamond p \rightarrow q$$  \hfill (2)  

$$p \leftrightarrow_d q \equiv (\Diamond p \rightarrow q) \land (\Diamond q \rightarrow \Diamond p)$$  \hfill (3)

With these connectives in place, he defines the theses of $D_2$ as:

The set of formula constructed using only the connectives $\neg, \land, \lor, \rightarrow_d$, and $\leftrightarrow_d$ such that when preceded by $\Diamond$ they become $M_2$ theses. [5]

In a trivial example,

$$D_21$$

$$p \rightarrow_d p \equiv \Diamond p \rightarrow p$$

(if it is possible that $p$ then $p$)

We see that though it is a $D_2$ thesis, it only becomes an $M_2$ thesis when preceded with a $\Diamond$. That is,
To understand why we hold $M_2$, we must understand how to simplify expressions with repeated modal operators. For this, we need to first decide which axiomatization of $M_2$ we will employ. We choose the $S_5$ axiomatization of $M_2$ which stipulates that:

$$XX...X□p \equiv □p \quad (4)$$

$$XX...X ◦ p \equiv ◦ p \quad (5)$$

Where each $X$ is either a $□$ or $◦$. With these axioms defined, we now just need laws for distribution of modal operators over implication, equivalence, and disjunction. Jaskowski defines the following 3 $M_2$ theorems for this purpose:

$$ ◦ (p \rightarrow_d q) \leftrightarrow ( ◦ p \rightarrow ◦ q) \quad (6)$$

$$ ◦ (p \leftrightarrow_d q) \leftrightarrow ( ◦ p \leftrightarrow ◦ q) \quad (7)$$

$$ ◦ (p \lor q) \leftrightarrow ( ◦ p \lor ◦ q) \quad (8)$$

With the $S_5$ axiomatization in mind, and these definitions provided, it’s easy to see that $◇(◇p \rightarrow p)$ reduces to $◇p \rightarrow ◦ p$. Which is clearly an $M_2$ thesis.

Using these new connectives we can show that modus ponens $(P, P \rightarrow_d Q \models Q)$ may be preserved if discussive implication is used in place of ordinary implication. As Jaskowski explains, this result depends on the $M_2$ theorem $◇(◇p \rightarrow q) \rightarrow ( ◦ p \rightarrow ◦ q)$ which follows from (5) and (6). The $D_2$ maintenance of modus ponens suggests that it may be more adept at modeling the discursive dilemma. The following section investigates that claim.

### 2.4 Applying $D_2$ to the Discursive Dilemma

Referring to Table 3, we see that $◇α$, $◇β$ and $◇(α \land β)$ are all true (and therefore $D_2$ validates each). Thus, the new “aggregation rule” that $D_2$ employs doesn’t produce the same inconsistent results when applied to the discursive dilemma. Referring to Table 4, we see that $α$ discursively implies $β$, similarly preventing the inconsistency if we use $\rightarrow_d$ in place of $\rightarrow$. This feature, along with its rejection of the implicational law of overfilling $(p \rightarrow_d (¬p \rightarrow_d q))$, suggests that $D_2$ handles the paradox more meaningfully. However, there is a certain sense in which information from the ballots is lost by applying this rule.

From a purely logical perspective, $D_2$ neatly resembles classical logic and facilitates inference in a judgment aggregation setting. But to see the practical limitations, consider the following two situations. In each, only a single discussant holds both $α$ and $β$: 
Table 5: A set of ballots where a single discussant validates both $\alpha$ and $\beta$

<table>
<thead>
<tr>
<th></th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\alpha \land \beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_1$</td>
<td>False</td>
<td>False</td>
<td>False</td>
</tr>
<tr>
<td>$d_2$</td>
<td>False</td>
<td>False</td>
<td>False</td>
</tr>
<tr>
<td>$d_3$</td>
<td>False</td>
<td>False</td>
<td>False</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$d_{99}$</td>
<td>False</td>
<td>False</td>
<td>False</td>
</tr>
<tr>
<td>$d_{100}$</td>
<td>False</td>
<td>False</td>
<td>False</td>
</tr>
<tr>
<td>$d_{101}$</td>
<td>True</td>
<td>True</td>
<td>True</td>
</tr>
</tbody>
</table>

Table 6: A different set of ballots where a single discussant validates both $\alpha$ and $\beta$

<table>
<thead>
<tr>
<th></th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\alpha \land \beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_1$</td>
<td>True</td>
<td>False</td>
<td>False</td>
</tr>
<tr>
<td>$d_2$</td>
<td>True</td>
<td>False</td>
<td>False</td>
</tr>
<tr>
<td>$d_3$</td>
<td>True</td>
<td>False</td>
<td>False</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$d_{49}$</td>
<td>True</td>
<td>False</td>
<td>False</td>
</tr>
<tr>
<td>$d_{50}$</td>
<td>True</td>
<td>False</td>
<td>False</td>
</tr>
<tr>
<td>$d_{51}$</td>
<td>False</td>
<td>True</td>
<td>False</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$d_{99}$</td>
<td>False</td>
<td>True</td>
<td>False</td>
</tr>
<tr>
<td>$d_{100}$</td>
<td>False</td>
<td>True</td>
<td>False</td>
</tr>
<tr>
<td>$d_{101}$</td>
<td>True</td>
<td>True</td>
<td>True</td>
</tr>
</tbody>
</table>

In the first set of ballots, we see that an overwhelming majority negate both $\alpha$ and $\beta$, and a single discussant affirms both. The second set is an expanded version of the discursive dilemma. Again, only a single discussant affirms both propositions, but the remaining discussants are evenly split between the two. Intuitively, these situations differ immensely. In the first, we can easily conclude that the majority holds neither $\alpha$ nor $\beta$ despite the lone dissenter. In the second, the disagreement is more prolific – there are two opposing factions and a lone compromiser.

Using $D_2$, however, this intuitive understanding is completely lost. In both situations, $\alpha$, $\beta$ and $\alpha \land \beta$ are possible (and therefore validated by $D_2$). While this leads to consistent aggregations, it does so at the expense of useful information – some of which can even be intuitively gleaned from the ballots. Thus, discursive implication and equivalence are fairly weak notions of their $L_2$ analogs – only that of possibility. If we instead consider the stronger notion of probability, we can obtain systems that more precisely capture the nature of democratic disagreement. This hints at the approach of strengthening the $\diamond$ operator. If we consider the modal operator defined to be $i$ out of $d$ discussants hold that or it is $i/d$ probable that, we may begin to unpack problem. The remainder of this paper will focus on extending $D_2$ using the unary modal operator $\diamond_{i/d}$ defined by (1) in the introduction.

\[^5\text{Sometimes I will use the notation } \diamond_\alpha \text{ where } \alpha \in [1/d, 1] \text{ for brevity.}\]
3 $D_2^\alpha$ and the Democratic Logics

3.1 Defining democratic implication and equivalence, $\to_{i/d}$ and $\leftrightarrow_{i/d}$

The modal operator $\diamondsuit_{i/d}$ is incredibly flexible – it encapsulates notions of possibility, necessity, and any fractional values in between for various choices of $i$. Following the example set by $D_2$ we may define connectives for democratic equivalence and democratic implication, $\leftrightarrow_{j/d}, \to_{j/d}$ as follows:

$$p \to_{j/d} q \equiv \diamondsuit_{j/d} p \to q$$

$$p \leftrightarrow_{j/d} q \equiv (\diamondsuit_{j/d} p \to q) \land (\diamondsuit_{j/d} q \to \diamondsuit_{j/d} p)$$

To facilitate using these connectives, we update (4) and (5) for our new operator:

$$XX...X \diamondsuit_{i/d} p \equiv \diamondsuit_{i/d} p$$

where each $X$ represents $\diamondsuit_{j/d} \forall i : 1 \leq i \leq d$

With these connectives’ behaviors well defined – we can generate logics similar to $D_2$ for specific choices of $i$. Before proceeding with that discussion, a notational clarification is useful. In this section, I referred to the modal probability operator using fractions $i/d$, but to the implicational/equivalence operators using fractions $j/d$. This choice was deliberate and is intimately related to the definition of a specific logic $D_2^{(i/d)}$. For now, simply think of $i$ as a reserved variable – its notational importance will be elucidated in the following section.

3.2 Defining the set $D_2^\alpha$

We can now form a family of democratic logics, defined by nearly the same conditions as Jaskowski’s $D_2$. Each logic in this family of logics consists of a set of formulae $\mathcal{T}$, theses of $D_2^{(i/d)}$ (for a specific choice of $i$), such that for each, the following holds:

1. $\mathcal{T}$ includes sentential variables and at most the following operators: $\neg, \land, \lor, \leftrightarrow, \to_{j/d}, \land$ and $\leftrightarrow_{j/d} \forall j \leq d$

2. Preceding $\mathcal{T}$ with the operator $\diamondsuit_{i/d}$ yields a theorem in the two-valued logic $M_2$ (using the axiomatization of $S_5$)

The importance of $i$ can now be stated more clearly. While the choice of $i$ defines a specific logic $D_2^{(i/d)}$, we may use connectives $\to_{j/d}$ and $\leftrightarrow_{j/d}$ for each $j \leq d$ (i.e. even for $j \neq i$). This facilitates the desired flexibility for describing judgment aggregation problems, preserving the maximum amount of information from an inconsistent deductive system. For clarity then, I will always use $i$ to refer to the choice of $\diamondsuit_{i/d}$ that may

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6 Sometimes $\leftrightarrow_{\beta}, \to_{\beta}$

7 Again, $\diamondsuit_{\alpha}$ and $\to_{\beta}, \leftrightarrow_{\beta}$ are often neater without being ambiguous. When the meaning is ambiguous, I use the verbose notation. Otherwise, I use the abbreviated form.
precede any $D_2^{(i/d)}$ thesis to produce an $M_2$ thesis. Any other variables ($j, k, l, \text{etc.}$) will refer to dummy variables, specific to the implication or equivalence relation they follow.

As the following analyses will show, with specific restrictions on the types of democratic implication and equivalence employed, various $L_2$ theses are either retained or invalidated. With access to a multitude of connectives, an informed choice engenders a much deeper sense of interpretability when modeling an inconsistent system.

To develop an intuition around the family of democratic logics described, the following section compares some $D_2$ theorems with their counterparts in $D_2^{(i/d)}$, and provide what restrictions (given the choice of $i$) are necessary. Before proceeding, it’s useful to consider the following theorems which help to reduce formulae:

$$\diamond j p \rightarrow \diamond k p$$  \hfill (12)

True $\forall k, j : 1 \leq k \leq j \leq d$

$$\neg \diamond h/d p \leftrightarrow \diamond j/d \neg p$$  \hfill (13)

True $\forall h, j : h + j > d$

The first is fairly straight forward and says that when at least $j$ discussants hold $p$, then the same is true for every $k$ less than $j$. The second is the law of distribution for negation over $\diamond i/d$. The $M_2$ theorem $\neg \Box p \leftrightarrow \diamond \neg p$, is a specific instance of this theorem for $j = 1$ and $h = d$. With these in mind, we can now begin the project of extending $D_2$ into $D_2^{(i/d)}$.

### 3.3 $D_2$ Methodological Theorems revisited

In the following sections, we will analyze a number of $D_2$ theses, presented originally by Jaskowski. To achieve this, understanding the three methodological theorems he offers, and their applicability in this new system is useful.

$D_2$ Methodological Theorem 1: Every thesis $T$ in $L_2$ that only contains the connectives $\rightarrow, \leftrightarrow, \vee$ becomes a thesis $T_d$ in $D_2$ when the implication symbols in $T$ are replaced by $\leftrightarrow_d, \rightarrow_d$.

Jaskowski leverages this theorem tremendously throughout his paper. However, his proof relies on the fact that for each variable in $T$, a substitution of the form $p/\diamond p$ occurs. In our system, unfortunately, in a single $D_2^{(i/d)}$ formula there may be many different operators $\rightarrow_{j/d}, \rightarrow_{k/d}$...etc. Thus, each switch $p/\diamond p$ is not necessarily guaranteed. However, the theorem holds if we stipulate that each instance of $\rightarrow$ and $\leftrightarrow$ is replaced with $\rightarrow_{i/d}$ and $\leftrightarrow_{i/d}$ for the specific $i$. By doing this we ensure that each formula has only one modal operator, $\diamond_{i/d}$, thus the substitution $p/\diamond_{i/d} p$ is valid. With this condition satisfied, the proof follows directly from Jaskowski’s paper.
**D₂ Methodological Theorem 2:** If \( T \) is a thesis in \( L₂ \) and includes at most the connectives, \( \land, \lor, \) and \( \neg \). Then:

1. \( T \)
2. \( \neg T \rightarrow Q \)

are theses of \( D₂ \).

This theorem holds in \( D₂ \) and in each \( D₂(i/d) \).

**D₂ Methodological Theorem 3:** If in a thesis \( T \) belonging to \( D₂ \), you replace each instance of \( \rightarrow d \) with \( \rightarrow \) and each instance of \( \leftrightarrow d \) with \( \leftrightarrow \), an \( L₂ \) thesis is obtained.

If we update this rule to say that we replace each instance of \( \rightarrow i/d \) and \( \leftrightarrow i/d \) in a \( D₂(i/d) \) thesis with \( \rightarrow \) and \( \leftrightarrow \), and further that the thesis contains no democratic implication or equivalence relations \( \rightarrow j/d, \leftrightarrow j/d \) such that \( j \neq i \), then it still holds.

**Proof:** Any \( M₂ \) thesis becomes an \( L₂ \) thesis once each modal operator is removed. By definition, \( D₂(i/d) \) theses when preceded with a \( \Diamond_i \) are \( M₂ \) theses. Expanding any instances of \( \rightarrow_i \) or \( \leftrightarrow_i \) only replaces them with \( \rightarrow \) and \( \leftrightarrow \) and the modal operator \( \Diamond_i \). Removing these yields an \( L₂ \) thesis. □

With these methodological theorems at our disposal, we can begin investigating \( D₂ \) theses under this new system.

### 4 Practical Applications of \( D₂(i/d) \)

The following sections are included to supplement intuition about \( D₂(i/d) \) and to examine some of its practical advantages. In the first two sections, I examine a selection of \( D₂ \) theses and non-theses offered by Jaskowski. Where possible, I give restrictions on the choices of connectives (relative to each other and the choice of \( i \)) that constrain these formulae to be \( D₂(i/d) \) theses. In the final section, I readdress the discursive dilemma and the examples offered in tables 5 and 6 to show that \( D₂(i/d) \) is capable of producing consistent results without sacrificing useful information about the inconsistent system of interest.

#### 4.1 \( D₂ \) Theses in \( D₂(i/d) \)

\[ D₂1 \]

\[ p \rightarrow_{j/d} p \]  \( (14) \)

Holds \( \forall i, j : 1 \leq i \leq j \leq d \)

Interestingly, even in this trivial case, the restriction on \( j \) is necessary. Though the result follows from (12), it’s helpful to examine it explicitly. Consider the system of 3 discussants in logic \( D₂(2/3) \) where \( j = 1 \). Here the formula reduces to \( \Diamond_{1/3}p \rightarrow \Diamond_{2/3}p \). Clearly, we may have that \( \Diamond_{1/3}p \) without having \( \Diamond_{2/3}p \), invalidating the formula.
With this result in mind, we can easily extend *modus ponens* for $D^{(i/d)}_2$. For clarity, I use the more modern presentation (rather than Jaskowski’s esoteric formulation).

**Modus Ponens**

\[ \mathcal{P}, \mathcal{P} \rightarrow_{j/d} Q \models Q \]  

\[ \text{Holds } \forall i, j : 1 \leq j \leq i \leq d \]

The first premise says that $i$ discussants hold $\mathcal{P}$. The second says that if $j$ discussants hold $\mathcal{P}$ then $i$ discussants hold $Q$. The conclusion says that $i$ discussants hold $Q$. Therefore by (14) the result follows.

The following can be shown by calculations – some examples are expanded for added clarity.

**$D_2^2$**

\[ (p \leftrightarrow_{j/d} q) \leftrightarrow_{k/d} (q \leftrightarrow_{l/d} p) \]  

\[ \text{Holds } \forall i, j, k, l : 1 \leq k \leq j \leq d, \text{ and } 1 \leq i \leq l \leq d \]

**$D_2^4$** (Law of contradiction)

\[ \neg(p \land \neg p) \]  

\[ \text{Holds by Methodological Theorem 2} \]

**$D_2^5$** (Conjunctonal Law of overfilling)

\[ (p \land \neg p) \rightarrow q \]  

\[ \text{Holds by Methodological Theorem 2} \]

**$D_2^6$**

\[ (p \land q) \rightarrow_{j/d} p \]  

\[ \text{Holds } \forall i, j : 1 \leq i \leq j \leq d \]

**$D_2^7$**

\[ p \rightarrow_{j/d} (p \land p) \]  

\[ \text{Holds } \forall i, j : 1 \leq i \leq j \leq d \]

**$D_2^8$**

\[ (q \land p) \rightarrow_{j/d} (p \land q) \]  

\[ \text{Holds } \forall i, j : 1 \leq i \leq j \leq d \]

**$D_2^9$**

\[ (p \land (q \land r)) \leftrightarrow_{j/d} ((p \land q) \land r) \]  

\[ \text{Holds } \forall i, j : 1 \leq i \leq j \leq d \]
\[ D_{210} \text{ (Law of importation)} \]

\[ (p \rightarrow_{j/d} (q \rightarrow_{k/d} r)) \rightarrow_{h/d} ((p \land q) \rightarrow_{l/d} r) \]  \hspace{1cm} (23)

Holds \( \forall i, h : 1 \leq i \leq h \leq d, \)
\( \forall j, k, l : \) when \( j + k \leq d \) then \( \min(j, k) \leq l \leq d \) or if \( j + k > d, \) then \( l > (j + k) - d \)

That is, the formula fails when:

**Case 1: \( i > h \)**

We may have \( \phi_{h/d} \) but not \( \phi_{i/d} \). So assume \( /p/ = /q/ = 1 \). By expanding each implication we have:

\( (\phi_{j/d} \rightarrow (\phi_{k/d} \rightarrow \phi_{h/d})) \rightarrow (\phi_{l/d} (p \land q) \rightarrow \phi_{i/d}) \)

Clearly if \( p, q \) are tautologies and \( \phi_{h/d} \) is true while \( \phi_{i/d} \) is false, the formula is false.

**Case 2: \( l \leq j + k - d \) (for \( j + k > d \))**

Consider \( d = 5, k = 4, j = 3, l = 1 \). Here \( j + k - d = 2 \) larger than \( l = 1 \)

From above we have:

\( (\phi_{3/5} \rightarrow (\phi_{4/5} \rightarrow \phi_{5/5} \rightarrow \phi_{h/5})) \rightarrow (\phi_{l/5} (p \land q) \rightarrow \phi_{i/5}) \)

Clearly, when \( r \) is a contradiction and we have a small minority that holds \( p \land q \) (i.e. \( \phi_{1/5} (p \land q) \) is true, but \( \phi_{3/5} p, \phi_{4/5} q \) is false), then the formula is invalidated.

**Case 3: \( l \leq \min(j, k)(j + k \leq d) \)**

Consider \( d = 5, k = j = 3, l = 1 \) From above we have:

\( (\phi_{2/5} \rightarrow (\phi_{2/5} \rightarrow \phi_{3/5} \rightarrow \phi_{h/5})) \rightarrow (\phi_{l/5} (p \land q) \rightarrow \phi_{i/5}) \)

As in Case 2, when \( r \) is a contradiction we may have a small minority such that \( \phi_{2/5} p, \phi_{2/5} q \) are false, but \( \phi_{1/5} (p \land q) \) is true, invalidating the formula

\[ D_{213} \]

\[ p \leftrightarrow_{j/d} \neg \neg p \]  \hspace{1cm} (24)

Holds \( \forall i, j : 1 \leq i \leq j \leq d \)

\[ D_{214} \]

\[ (\neg p \rightarrow_{j/d} p) \rightarrow_{k/d} p \]  \hspace{1cm} (25)

Holds \( \forall i, j, k : 1 \leq i \leq k \leq d, \) and \( (i + j) \leq d + 1 \)

**Case 1: \( k < i \)**

Assume \( k = 1, i = 2, d = 3 \).

Clearly we may have \( \phi_{1/3} p, \phi_{2/3} \neg p, \) but not \( \phi_{2/3} p. \)

Thus the formula is invalid.

\[ 12 \]
Case 2: \( i + j > d + 1 \)

Consider \( d = 5, j = 4, i = 3 \). Here \( i + j = 7 > 5 + 1 \).
If only 2 discussants hold that \( p \). Then \( \Diamond_{4/5} \neg p \) false, and therefore 
\( \neg p \to_{4/5} p \) is true.
But we do not have that \( \Diamond_{3/5}p \), only \( \Diamond_{2/5}p \)
Thus the formula is invalid.

\[ D_{215} \]
\[
(p \to_{j/d} \neg p) \to_{k/d} \neg p
\]
Holds \( \forall i, j, k : 1 \leq i \leq k \leq d \), and \( (i + j) \leq d + 1 \)

\[ D_{216} \]
\[
(p \leftrightarrow_{j/d} \neg p) \to_{k/d} p
\]
Holds \( \forall j, k : 1 \leq j \leq d \), and \( 1 \leq k \leq d \)

\[ D_{217} \]
\[
(p \leftrightarrow_{j/d} \neg p) \to_{k/d} \neg p
\]
Holds \( \forall j, k : 1 \leq j \leq d \), and \( 1 \leq k \leq d \)

\[ D_{218} \]
\[
(p \to_{j/d} (q \land \neg q)) \to_{k/d} \neg p
\]
Holds \( \forall i, j : (i + j) \leq d + 1 \)

\[ D_{219} \]
\[
(-p \to_{j/d} (q \land \neg q)) \to_{k/d} p
\]
Holds \( \forall i, j : (i + j) \leq d + 1 \)

\[ D_{220} \]
\[
\neg (p \leftrightarrow_{j/d} \neg p)
\]
Holds \( \forall j : 1 \leq j \leq d \)

\[ D_{221} \]
\[
(p \to_{j/d} q) \to_{k/d} p
\]
Holds \( \forall i, j : 1 \leq i \leq j \leq d \) and \( 1 \leq k \leq d \)

\[ D_{222} \]
\[
\neg (p \to_{j/d} q) \to_{k/d} \neg q
\]
Holds \( \forall i, j, k : 1 \leq i \leq k \leq d \) and \( 1 \leq j \leq d \)

\[ D_{223} \]
\[
p \to_{j/d} (\neg q \to_{k/d} \neg (p \to_{i/d} q))
\]
Holds \( \forall i, k, l : 1 \leq i \leq k \leq d \), and \( 1 \leq l \leq i \leq d \)
**Case 1: **$k < i$

$D_226$ reduces to:

\[ \diamondsuit_{j/d} p \rightarrow (\diamondsuit_{k/d} \neg q \rightarrow \diamondsuit_{l/d} (\diamondsuit_{l/d} p \rightarrow q)) \]

Choosing $k = 1$, $i = 2$, $d = 3$, produces:

\[ \diamondsuit_{j/d} p \rightarrow (\diamondsuit_{1/3} \neg q \rightarrow \diamondsuit_{2/3} (\diamondsuit_{l/d} p \rightarrow q)) \]

We see that when $\diamondsuit_{1/3} \neg q$ is true, $q$ is either false or held false by a single discussant (i.e. $\diamondsuit_{2/3} \neg q$ is false) When $p$ is a tautology, and $\diamondsuit_{1/3} \neg q$ is true, we then have:

$\diamondsuit_{2/3} (\diamondsuit_{l/d} p \rightarrow q)$ false, but

$\diamondsuit_{1/3} (\diamondsuit_{l/d} p \rightarrow q)$ true

Invalidating the formula.

**Case 2: **$i > l$

Assume $l = 1$, $i = 2$, $d = 3$ From above we have: $\diamondsuit_{j/d} p \rightarrow (\diamondsuit_{k/d} \neg q \rightarrow \diamondsuit_{2/3} (\diamondsuit_{l/d} p \rightarrow q))$

Consider the case when a lone discussant holds that $p$, and all hold $q$ false.

Then we have $\diamondsuit_{1/3} p$ true, but not $\diamondsuit_{2/3} p$. Therefore:

$\diamondsuit_{2/3} (\diamondsuit_{l/d} p \rightarrow q)$ is false, invalidating the formula.

Intuition for this problem comes from equation (13)

### 4.2 Non-$D_2$ Theses with the new implication rule

(\textit{nonD}_2)1

\[ p \rightarrow_{j/d} (q \rightarrow_{k/d} (p \land q)) \] \hfill (35)

Holds $\forall i, j, k : 1 \leq j + k - d \leq i \leq d$

(\textit{nonD}_2)2 Law of exportation:

\[ ((p \land q) \rightarrow_{j/d} r) \rightarrow_{k/d} (p \rightarrow_{l/d} (q \rightarrow_{m/d} r)) \] \hfill (36)

Fails for all assignments of $i, j, k, l, m$

(\textit{nonD}_2)3 Implicational law of overfilling:

\[ p \rightarrow_{j/d} (\neg p \rightarrow_{k/d} q) \] \hfill (37)

Holds $\forall j, k : j + k > d$

(\textit{nonD}_2)3a

\[ p \rightarrow_{j/d} (\neg p \rightarrow_{k/d} \neg q) \] \hfill (38)

Holds $\forall j, k : j + k > d$

(\textit{nonD}_2)6

\[ (p \rightarrow_{j/d} \neg p) \rightarrow_{k/d} ((\neg p \rightarrow_{l/d} p) \rightarrow_{h/d} q) \] \hfill (39)

Holds $\forall j, k, l, h : j + k > d$, and $l + h > d$
$$(\text{nonD}_2)6$$

$$(p \to j/d \neg p) \to k/d ((\neg p \to l/d p) \to h/d (p \land \neg p))$$  (40)

Holds $\forall j, k, l, h : j + k > d$, and $l + h > d$

$$(\text{nonD}_2)7$$

$$-(p \to j/d p) \to k/d q$$  (41)

Fails $\forall j, k$

$$(\text{nonD}_2)8$$

$$(p \to j/d q) \to k/d (\neg q \to l/d \neg p)$$  (42)

Holds $\forall i, j, k, l : j + i > d$, and $k + l > d$

$$(\text{nonD}_2)9$$

$$-(p \to j/d \neg q) \to k/d (q \to l/d p)$$  (43)

Holds $\forall i, j, k, l : j + i > d$, and $k + l > d$

$$(\text{nonD}_2)10$$

$$(p \to j/d q) \to k/d ((p \to l/d \neg q) \to h/d \neg p)$$  (44)

Holds $\forall i, j, k, l, h : 1 \leq l \leq j \leq d$, and $j + i > d$, and $k + h > d$

$$(\text{nonD}_2)11$$

$$-(p \to j/d q) \to k/d ((\neg p \to l/d \neg q) \to h/d p)$$  (45)

Holds $\forall i, j, k, l, h : 1 \leq l \leq j \leq d$, and $j + i > d$, and $k + h > d$

Though some theses presented in Jaskowski’s original paper have been omitted (as evidenced by the gaps in the numbering), the ones presented here adequately cover the topics expressed by those.

4.3 Applying $D_2^{(i/d)}$ in the previous scenarios

If we now reconsider the situations presented in tables 5 and 6 (used to describe the limitations of $D_2$) using a logic in $D_2^{\alpha}$, we see that we can retain desirably consistent results without the same information loss. In each, $\alpha \land \beta$ only holds in $D_2^{(1/101)}$, but in the former both $\alpha$ and $\beta$ also only hold in $D_2^{(1/101)}$. In the latter, however, $\alpha$ and $\beta$ both hold in $D_2^{(51/101)}$. For these simple scenarios, a complex logical model seems unwieldy (and perhaps unnecessary), but in a more nuanced example the advantage of $D_2^{\alpha}$ is more immediate. Consider the following two sets of ballots, analogous to those presented in tables 5 and 6. In the first a simple majority rule correctly suggests that the voters hold $\alpha$, but neither $\beta$ nor $\alpha \to \beta$. The second is another expanded example of the discursive dilemma – this time using implication.
<table>
<thead>
<tr>
<th></th>
<th>α</th>
<th>β</th>
<th>α → β</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>True</td>
<td>False</td>
<td>False</td>
</tr>
<tr>
<td>2</td>
<td>True</td>
<td>False</td>
<td>False</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>100</td>
<td>True</td>
<td>False</td>
<td>False</td>
</tr>
<tr>
<td>101</td>
<td>True</td>
<td>True</td>
<td>True</td>
</tr>
</tbody>
</table>

Table 7: A set of ballots where α discussively implies β

<table>
<thead>
<tr>
<th></th>
<th>α</th>
<th>β</th>
<th>α → β</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>True</td>
<td>True</td>
<td>True</td>
</tr>
<tr>
<td>2</td>
<td>True</td>
<td>True</td>
<td>True</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>50</td>
<td>True</td>
<td>True</td>
<td>True</td>
</tr>
<tr>
<td>51</td>
<td>True</td>
<td>False</td>
<td>False</td>
</tr>
<tr>
<td>51</td>
<td>False</td>
<td>False</td>
<td>True</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>100</td>
<td>False</td>
<td>False</td>
<td>True</td>
</tr>
<tr>
<td>101</td>
<td>False</td>
<td>False</td>
<td>True</td>
</tr>
</tbody>
</table>

Table 8: An alternate set of ballots where α discussively implies β

In $D_2$, we have that in each set of ballots $α \rightarrow_d β$. Just as before, the intuition provided by a simple majority rule fails to differentiate the two situations. If we consider these ballots using the more general system provided above, we can easily see that in the first $α \rightarrow_{1/101} β$ is true for $D_2^{(1/101)}$ (this is equivalent to saying it holds in $D_2$). In the second, however, we can say that $α \rightarrow_{51/101} β$ in $D_2^{(51/101)}$, $\forall i : 1 \leq i \leq 50$ – capturing substantially more information about the system in question. This is exactly the kind of flexibility that the set $D_2^α$ is capable of providing.

5 Summary and Future Research

5.1 Modal Probability Logic

Before concluding, it’s worth mentioning the similarities between the work presented here as $D_2^α$ and the so-called modal logic of probability presented by Heifetz and Mongin in 1998 [3]. To facilitate the axiomatization of this logic, they introduce the belief operator $L_α$ defined for rational $α \in [0, 1]$, interpreted as “the probability is at least $α$”. In this way, they extend modal logic to consider mappings from possible worlds to probability measures, rather than sets. Using $L_α$, they go on to define other similar operators $M_α$ ($M_αψ \leftrightarrow L_1 - α \neg ψ$), $E_α$ ($E_αψ \leftrightarrow L_α ψ \land M_αψ$), interpreted as “the probability is at most $α$” and “the probability is exactly $α$” respectively. The modal operator I have introduced
\( \diamond i/d \) (used to define a logic \( D_2^{(i/d)} \)) captures the exact same notion as \( L_\alpha \), if we consider the proportion of discussants that hold \( p \) as the probability of \( p \). Their model-theoretic approach differs strongly from mine, but suggests an avenue for further research. If (as I suspect) \( \diamond_\alpha \) is semantically equivalent to \( L_\alpha \), then a great deal more can easily be said about \( D_2^{(\alpha)} \) based on their work. This exceeds the scope of this paper, but is an interesting topic for further research.

### 5.2 Final Remarks

As the previous sections hopefully have made evident, the family of logics, \( D_2^{(\alpha)} \), are particularly adept at modeling systems of rational discourse without losing information to over-simplification. I claim that this makes them useful for the practical analysis of judgment aggregation problems. Though they cannot resolve the well-studied impossibility results posed by such problems, an intelligent choice of \( i \) and use of connectives guarantee consistent aggregation results without reducing those results to incredibly weak statements. Thus, this extension of \( D_2 \) is pragmatically (rather than semantically) driven. That is, while \( D_2 \) disregards information in the pursuit of logically “neat” results, \( D_2^{(\alpha)} \) maintains those same results without sacrificing potentially elucidatory information. Though the logic itself is somewhat “messier”, the practical benefits are significant.

### References


