Math Logic II: Gödel’s Incompleteness Results
To be given in the fall of 2020

Course Description: This is a rigorous derivation of Gödel’s incompleteness results and additional basic results related to them: Definitions and theorems about computable functions and relations, and some non-computability results that follow from Gödel’s technique. These are among the most philosophical significant mathematical theorems ever proved. Enrolled students should have a sufficient mastery of first-order logic; but this is not a pre-condition, because what is required is not much and will be learned in the course itself. A certain technical ability is however required. The course had been previously cross listed with Computer Science; the cross-listing has been omitted due to formal regulations of the registrar. Yet it is the same course and it is recognised by the math department and CS as fulfilling the requirement for electable courses. The last time this course was offered (in 2017) some undergraduates in CS did extremely well.

We begin the course by tracing the way by which Gödel might have arrived at his idea of getting a sentence that says about itself “I am not provable”. We shall also discuss some historical evidence pointing out von Neumann’s role in the process. Thus, we prove the first incompleteness theorem, assuming that some arithmetization of the language is available and then proceed to provide the most transparent way of getting the arithmetization. At the end we shall also give the general idea underlying the second incompleteness theorem, and go through the proof if time allows.

The course relies on course notes that have been upgraded during the years, which now amount to 5 chapters of a small textbook, including exercises. (There is also an additional chapter which supplies an overview of first order logic, but all that is needed will actually be taught in class).

The final grade will be determined mostly by a final take-home exam, as well as homework, which consists of some of the exercises from the notes. Usually there is also a weight assigned to participation in class, but in the present situation it is at present impossible to assign weights.

Syllabus

The following syllabus is rather comprehensive. Depending on the enrolled students, some items might be covered at greater length than others. And some items might be skipped. This happened in fact in past years.

1. An intuitive guide to Gödel's proof. Cantor's diagonalization. Richard's Paradox. Reconstructing the Liar paradox within a formal system that contains a truth predicate.

   1.1. Some historical background. Cantorian set theory. The paradox that arises when we treat the universal class as a set. Hilbert's program and Hilbert's idea of proving the consistency of a theory not by providing an interpretation for it, but by reasoning about the formal deductive system. Gödel's results implied that the goals of the program are unachievable.

2. The basic form of Gödel's project: arithmetization of the language (encoding the language into the natural numbers) and using an adequate theory to represent various syntactic relations and functions. The diagonal function. The representation of proofs by natural numbers. The Gödel sentence, γ, which says about itself "I am not provable". Gödel's first incompleteness theorem.

   The fixed point theorem (aka the self-reflection theorem): For every wff $\psi(v)$, there is a sentence σ such that $T \vdash \sigma \leftrightarrow \psi(\ulcorner\sigma\urcorner)$.

   2.1. Constructing sentences that "speak about themselves" in natural language.

3. The first-order language of arithmetic, based on addition multiplication and the successor function. The *standard model* of natural numbers and its philosophical significance (Kronecker: "God created the natural numbers all the rest is the work of man"). Extending the language by adding auxiliary predicates and function symbols that are definable by wffs. Bounded quantification. The μ operator. Encoding pairs and triples of natural numbers by natural numbers. Encoding all finite sequence of natural numbers into natural numbers, by using the Chinese remainder theorem. The use of that apparatus in order to express the usual inductive definitions of arithmetical functions.

4. A transparent Gödel numbering of the language, in which wffs and terms are represented by trees that fully display the syntactic structure. The coding uses only wffs based on sentential connectives, bounded quantification and the μ operator. Defining in this language the diagonal function.

5. Computable (aka *recursive*) functions and relations. The problem of characterizing the intuitive notion of "computable" or "effectively given". Gödel's basic idea was that, under the arithmetization of the language, all syntactic relations and functions should be computable. We shall establish the basic results concerning computable functions and relations, as well as the notion of *computably enumerable* (aka *recursively enumerable*). Basic results such as: a set is computable iff the set and its complement are computably enumerable.

   5.1. Computable functions and relations via Turing machines (very short summary)
   5.2. Primitive recursive functions and relations and the Ackerman function.

6. (I) Peano's Arithmetic, $\mathbb{P}$, and its basic properties. The arithmetic hierarchy. (II) Robinson's theory $\mathbb{Q}$. This is a much weaker theory, based on seven axioms, without an induction schema. All computable functions and relations are representable in $\mathbb{Q}$.

7. The Church-Turing theorem that claims that first order logic is not decidable, that is: the set of all theorem of pure first-order logic is not computable. In other words this means that there is no computer program that, for any given sentence, σ, gives us the correct answer to the question: is σ a theorem of FOL (pure first-order logic)? There are various versions of this theorem, because the decidability of FOL can depend on the non-logical vocabulary of the language. The strongest result (which we shall not prove) is this: if the language contains a binary relation symbol (i.e., a 2-place predicate) then FOL is not decidable.

8. The *Rosser sentence*: a sentence, ρ, for which the following holds: If T is consistent then (i) ρ is not provable in T and (ii) $\neg\rho$ is not provable in T. This improves Gödel's first incompleteness result: the Gödel sentence γ satisfies: (i) if T is consistent then γ is not provable, (ii) if T is ω-consistent then $\neg\gamma$ is not provable. As remarked in the discussion of the first incompleteness theorem, ω-consistency is stronger than consistency.

9. Gödel second incompleteness theorem, provability predicates and Löb's theorem.

   Gödel's second incompleteness theorem claims that if a theory is "sufficiently strong", so that it can describe its own syntax, and certain deductive reasoning is expressible in it, then, if T is consistent, the consistency of T is not provable in T. The consistency of T, CON_T, can be expressed in various ways, for example, by saying that there is an unprovable sentence, or by saying that a particular sentence, say '$0\neq0$' is not provable in it. Gödel's intuitive argument assumes that T is sufficiently strong so that the proof of the first claim in the first incompleteness theorem ("If T is consistent then γ is not provable in T") can be formalized within T. An analysis of the proof shows that the wff $Prv_{\mathrm{T}}(x)$, which says that x is provable in T, should satisfy certain conditions. A wff that satisfies these condition is called a *provability predicate*.

In 1952 Henkin asked what is implied, if anything, with regard to a sentence, η, that says about itself that it *is* provable: $\mathrm{T} \vdash \eta \leftrightarrow Prv_{\mathrm{T}}(\ulcorner \eta \urcorner)$? In 1955 Löb gave a surprising answer to that question:

If $\mathrm{T} \vdash Prv_{\mathrm{T}}(\ulcorner \eta \urcorner) \rightarrow \eta$, then $\mathrm{T} \vdash \eta$.

Gödel's second incompleteness theorem can be easily deduced from Löb's theorem.

10. The question whether the wff $Prv_{\mathrm{T}}(x)$ is a provability predicate in the case where $\mathrm{T} = \mathbb{P}$ (i.e., Peano's arithmetic) remains to be answered. The answer is positive. In fact, this can be established in a finitely axiomatizable sub-theory of $\mathbb{P}$, and it implies that Gödel's second incompleteness theorem applies: The proof is somewhat involved but a short intuitive sketch can be given.

11. Various generalizations of the fixed point theorem.

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