Probabilistic Semantics for Modal Logic

By

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Abstract

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We develop a probabilistic semantics for modal logic, which was introduced in recent years by Dana Scott. This semantics is intimately related to an older, topological semantics for modal logic developed by Tarski in the 1940's. Instead of interpreting modal languages in topological spaces, as Tarski did, we interpret them in the Lebesgue measure algebra, or algebra of measurable subsets of the real interval, [0, 1], *modulo sets of measure zero*. In the probabilistic semantics, each formula is assigned to some element of the algebra, and acquires a corresponding probability (or measure) value. A formula is satisfed in a model over the algebra if it is assigned to the top element in the algebra—or, equivalently, has probability 1.

The dissertation focuses on questions of *completeness*. We show that the propositional modal logic, $S4$, is sound and complete for the probabilistic semantics (formally, $S4$ is sound and complete for the Lebesgue measure algebra). We then show that we can extend this semantics to more complex, multi-modal languages. In particular, we prove that the dynamic topological logic, $S4C$, is sound and complete for the probabilistic semantics (formally, $S4C$ is sound and complete for the Lebesgue measure algebra with O-operators). The connection with Tarski's topological semantics is developed throughout the text, and the frst substantive chapter is devoted to a new and simplifed proof of Tarski's completeness result via well-known fractal curves.

This work may be applied in the many formal areas of philosophy that exploit probability theory for philosophical purposes. One interesting application in metaphysics, or mereology, is developed in the introductory chapter. We argue, against orthodoxy, that on a 'gunky' conception of space—a conception of space according to which each region of space has a proper subregion—we can still introduce many of the usual topological notions that we have for ordinary, 'pointy' space.

To Dana and Grisha,
for turning a philosopher into a mathematician,

and to Barry and Paolo,
for turning her back again.

Acknowledgements

This dissertation had its beginnings in the Colorado mountains. I went there in a break between Summer Session 2009 and the beginning of the fall semester to visit a friend, Darko Sarenac. At the time, I was having serious doubts about fnishing my degree, and was exploring the possibility of dropping out to become a photographer in the more remote parts of Southwestern New Mexico. Darko and I discussed the different things I might photograph, and even, as I remember, made plans to travel together with my camera to Wyoming and the South.

Soon enough, though, we got to talking about logic. On a hike up a mountain as a storm set in, I learned that there was a deep connection between topology and modal logic—indeed, that there was a whole feld called *topological modal logic* that had been quite active at Stanford and elsewhere in the last several years. In those 48 hours, Darko taught me the basics. By the time he dropped me off at the airport in Denver, we had plans to write a paper together. (This paper forms the frst chapter of the dissertation.) Thank you, Darko. Without you, I would be somewhere in New Mexico.

I want to thank, most of all, the people who have worked with me throughout my graduate career at Berkeley. I was incredibly fortunate to have Barry Stroud as an advisor from very early on. Our conversations shaped the way that I think about so many things, and his way of doing philosophy has been a great infuence on me. Although the topic I eventually chose for my dissertation was quite far afeld from Barry's own interests, he was the frst to encourage it. More than anyone else, Barry was witness to the many ups and downs of my career at Berkeley, and I always felt his staunch support and confdence.

I was also very fortunate to have Paolo Mancosu as an advisor and mentor. Paolo was the frst professor to take me on as a Graduate Student Instructor. I learned from him how logic could be taught in a way that was clear, engaging, and philosophically rich. When it came time to my writing a dissertation in modal logic, Paolo was aware of the many professional challenges that lay ahead and despite this, was fully supportive of the work I was doing and the project I had chosen. He provided me with invaluable advice and help at critical moments.

In the frst few days of working together, Darko surreptitiously sent an e-mail to Grigori Mints at Stanford, encouraging him to be in contact with me. I still remember seeing Grisha for the frst time after many years at a talk by Dana Scott in the Berkeley Logic Colloquium. At the talk, Dana introduced a new, probabilistic semantics for modal logic—a semantics about which very little was known at the time. Some days after the talk, Grisha approached me. "Tamar," he said, "Vai you not prove completeness?" in his inimitable accent. Grisha became a mentor to me of the best kind, always pointing me in the way of interesting questions, and giving practical advice at every turn in the road. I am very grateful for his taking me in with such generosity and kindness.

I am indebted on so many scores to Dana Scott. First, for introducing the probabilistic semantics: the semantics which this dissertation is about. What beautiful defnitions and ideas! Second, for taking me on as an informal student, once I had begun working on a completeness proof, as Grisha had suggested. But most of all, for his generosity and keen insights.

I have enjoyed great conversations with Justin Bledin, Branden Fitelson, Peter Koellner, Mike Martin, Christos Papadimitriou, James Stazicker, Seth Yalcin, and many others. I want to thank my sister and parents, for the support—and tremendous endurance!—they showed throughout this long endeavor.

Contents

Chapter 1

Introduction

1.1 Introduction

Almost half a century has now gone by since S. Kripke introduced Kripke semantics for modal logic. This semantics crystalized ideas in the analysis of modal propositions that can in some sense be traced back to Leibnitz, and his conception of 'necessity' as that which holds not just in the *actual* world, but in all *possible* worlds. Today Kripke semantics is standard not just in philosophical circles, but in such related disciplines as linguistics, computer science, and mathematics. No other semantics for modal languages rivals the simplicity and fexibility of the Kripke framework.

But long before Kripke, there was Tarski.

Looking at the axioms for the modal logic, $S4$, Tarski realized that, rearranged a certain way, these axioms resembled the axioms used by mathematicians to describe a *topological space*.[1] If you are unfamiliar with topology, don't worry. Think of a topological space (or simply a *space*) as a collection of points glued together in some way. The most familiar space is, perhaps, three-dimensional Euclidean space. Here we think of individual points as triples of real numbers. This space has some special features: between every two points, there is a well-defned distance; a sequence of points that converges, converges to a single point; and so on. What Tarski showed is that modal logic can be interpreted in topological spaces, and that—in a sense to be further specifed below—the modal logic $S4$ is the logic of topological spaces. Here, rather than thinking of the 'necessity' or 'D'-modality as picking out some collection of possible worlds, Tarski thought of it as

[1]Recall that a topological space is a pair, X, , where X is a set, and is a collection of subsets of X that is closed under fnite intersections, arbitrary unions, and contains the entire set X and the empty set, $\emptyset$.

a *spatial* operator, which picks out the *interior* of a region of topological space.

Tarski and McKinsey's work in the 1930's and 1940's led to what is now called the topological semantics for modal logic. Their elegant completeness results predate Kripke semantics by more than a decade, but in the years after the introduction of the Kripke framework, the topological semantics was largely forgotten. The fexibility of the Kripke framework—the fact that it can be used to model not just $S4$, but many different propositional and predicate modal logics—as well as its intuitive appeal are perhaps jointly responsible for the near-oblivion into which the topological semantics fell. In the last ffteen years or so, however, things have changed. Modal logicians, familiar with the many advances in temporal logics (or modal logics used to describe time, and temporal processes) started asking, 'What about a modal logic of space?' Tarski's work on the topological semantics came to be seen as the foundation stone of a much broader project: using modal logic to describe, make distinctions between, and systematize our reasoning about space and spatial structures. This research program has produced many new and interesting results in recent years: logicians have simplifed and refned Tarski and McKinsey's original completeness results; extended Tarski's topological semantics to more complex, multi-modal languages; and proved new results concerning the model theory and complexity of these extensions.

In the pages that follow, we take Tarski's topological semantics as our starting point. This is not to say that we ignore Kripke's relational semantics—far from it. Interesting relationships between the two will be developed throughout the text. But the primary aim of this work is not, in fact, to develop *either* Tarski's topolog- ical semantics, or Kripke semantics. Rather, it is to introduce the reader to a new way of interpreting modal languages—one that can be developed quite naturally, as we'll see, from Tarski's topological semantics, but which differs in important ways from any of the well-known semantics for modal logics to date.[2] Those semantics all share the following feature. In a given modal model (or formal interpretation of the modal language), each formula is either *true* or *false*. In Kripke semantics, we say that a formula is true in a model if it's true at every (accessible) possible world in the model. In the topological semantics, we say that a formula is true in a model if it's true at every point in the relevant topological space. What if instead we inter- preted modal languages probabilistically? What if, in other words, each formula in a given model got assigned not just a *truth* value, but a *probability* value between 0 and 1? The idea for a probabilistic semantics was introduced by Dana Scott in the last several years, in talks given at Stanford and Berkeley. As Scott said, the semantics "provides rich ingredients for building many kinds of structures having

[2]For a probabilistic semantics for classical logic, see K. Popper's (31) and H. Field's (12). See also Keisler's (16) and (17).

non-standard random elements." At the time, however, many fundamental questions about the semantics—particularly those relating to completeness—were still unanswered. In the chapters that follow, we answer some of these questions, and show that the probabilistic semantics can be elegantly extended to more complex, multi-modal languages.

In embarking on the work that follows, the question naturally arises: Why defne a new semantics for modal logic in the frst place? Isn't the standard Kripke semantics good enough?

There are two ways to respond. On the one hand, we may start out from an interest in existing modal languages (or existing axiomatic modal systems), and be interested in what the different semantics for these languages are. Here, of course, the probabilistic semantics will have quite different features from the standard Kripke semantics and even from the topological semantics for $S4$. Formulas, as we noted, acquire not just truth values in probabilistic models, but probability values. Someone interested in the various uses to which probability theory has been put in the more formal areas of philosophy might take interest in this new semantics for this reason. But secondly, one might start out from an interest in certain mathematical objects themselves—topological spaces, say, or topological spaces together with Borel measures in the present case. Then one will want to know: to what extent can modal languages describe, make distinctions between, and help us reason about these structures? From this point of view, the fexibility of Kripke semantics—the fact that it can be used to interpret not just $S4$, but many different modal logics—is not essential. What we want to know is what modal logics the mathematical objects we're interested in give rise to, and what distinctions between such objects can be made within the confnes of different modal languages.[3]

As the reader moves forward through the work of the next chapters, she is invited to keep these two perspectives in mind. The new semantics presented here is not meant as a *rival* for Kripke (or relational) semantics. Rather, the hope is that the probabilistic semantics can take its place alongside those other semantics,

[3]J. Van Benthem makes this point in connection with Tarski's topological semantics:

> Some modal logicians see topological models as a means of providing new semantics for existing modal languages, mostly for logic-internal purposes. This can be motivated a bit more profoundly by thinking of topologies as models for information, making this interest close to central logical concerns. But someone primarily interested in Space as such will not worry about the semantics of modal languages. She will rather be interested in spatial structures by themselves, and spatial logics will be judged by how well they analyze old structures, discover new ones, and help in reasoning about them. (40, p. 11)

opening up some new avenues, both philosophically and mathematically. Why not let a thousand fowers bloom?

1.2 Modal beginnings

[4]

But frst: what exactly is modal logic?

The standard propositional modal language consists of some countable collection of propositional variables, $\{P_n \mid n = 1, 2, 3, \ldots\}$, the Boolean connectives, $\{\neg, \vee, , \rightarrow, \leftrightarrow\}$, and the two modal symbols, D and 3. The symbols D and 3 are typically interpreted as expressing 'It is necessary that . . . ' and 'It is possible that . . . ,' respectively. More generally, modal symbols may be used to express a host of modalities from natural language—including, as we'll see, temporal, deontic, epistemic and, of course, metaphysical modalities. What exactly *is* a modality? R. Goldblatt says,

> A modality is any word or phrase that can be applied to a given statement S to create a new statement that makes an assertion about the mode of truth of S: about when, where or how S is true, or about the circumstances under which S may be true. (13, p. 310)

Goldblatt gives as examples the English language expressions, "henceforth," "eventually," "hitherto," "previously," "it is obligatory/forbidden/permitted/unlawful that," "it is known to X that," "it is common knowledge that," "it is believed that," and so on. Modal logics, we can say, are logics expressed in modal languages. They have been used to get at the meaning of, and formalize many of these English-language modalities.

1.2.1 Early motivations

The modern history of modal logic begins perhaps with C.I. Lewis. Lewis was motivated by the idea of understanding the English-language "implies"—a conditional connective that he took to have quite different properties from the material conditional of classical logic. "Expositors of the algebra of logic," Lewis noted, "have not always taken pains to indicate that there is a difference between the algebraic and ordinary meanings of implication." Lewis was particularly disturbed by what have come to be known as the paradoxes of the material conditional: the

[4] My account of the history here follows Goldblatt in (13). See his excellent discussion for much more detail.

fact that in classical logic a false proposition implies (in the algebraic sense) any proposition, and a true proposition is implied by any proposition. Insymbols,

$$\neg P \rightarrow (P \rightarrow Q);$$

$$P \rightarrow (Q \rightarrow P)$$

Under the ordinary meaning of implication, Lewis thought that 'P implies Q' means something like, 'Q can be legitimately inferred from P.' But one cannot legitimately infer any proposition from a false proposition. The paradoxes of the material conditional highlighted the way in which the material conditional of classical logic failed to capture the ordinary meaning of "implies"—a connective which Lewis thought stood at the foundations of fundamental notions in logic. "Unless 'implies' has some 'proper' meaning, there is no criterion of validity, no possibility even of arguing the question whether there is one or not," Lewis claimed. "And yet the question, What is the 'proper' meaning of 'implies'? remains peculiarly diffcult." (24, p. 325)

What system of logic, if not the classical one, could formalize the ordinary meaning of "implies"? The proposition expressed by 'A implies B' was, according to Lewis, equivalent to the proposition expressed by 'Either not-A or B.' But Lewis distinguished between what he called an extensional and intensional reading of "or." On the extensional reading, "or" is the truth-functional disjunction of classical logic. This yields the algebraic meaning of "implies" as a material conditional. But on the intensional reading of "or," Lewis claimed that "at least one of the disjuncts is *necessarily* true." Using this intensional reading to understand the ordinary meaning of implies, 'A implies B' is equivalent to 'Necessarily not-A or B." To understand the ordinary "implies," Lewis was moved to appeal to modal vocabulary—vocabulary that he thought functioned differently from any of the truth-functional connectives of classical logic.

Lewis came at modal logics from a syntactic, or axiomatic, point of view. His aim was to identify axioms and rules of inference in a new, modally-enriched language—ones that would be appropriate to what he took to be ordinary implication. In an appendix to their 1932 volume, *Symbolic Logic*, Lewis and Langford defned fve different axiomatic modal systems, $S1$ - $S5$. In these systems '3' is taken as a modal primitive, with the intended interpretation "possibly" or "it is possible that." The strict conditional—which was meant to formalize ordinary implication—is then defned in terms of this modal primitive as follows:

$$P \Rightarrow Q \equiv \neg 3 (P \,\&\, \neg Q)$$

In words: 'P implies Q' is equivalent to 'It is not *possible* that P and not-Q.' (Although Lewis did not himself introduce a separate "necessity" operator, D, it can be defned in terms of 3 and $\neg$ in the usual way: $\mathrm{D}\varphi \equiv \neg 3 \neg\varphi$. In words: ‘Necessarily P’ is equivalent to ‘It is not possible that not-P.’) The systems, *S*1 - *S*5, were the frst modern axiomatic systems of modal logic.

1.2.2 Relational semantics for modal languages

Already at the beginning, there was a range of views about what modalities the symbols ‘D’ and ‘3’ naturally expressed. While Lewis took them to symbolize necessity and possibility, respectively, Gödel saw in the new language a way of talking about *provability* within a formal system. He interpreted ‘D’ as the sentential operator ‘It is provable that...,’ and argued that on this interpretation, *S*4 was the correct axiomatic system. McKinsey and Tarski, meanwhile, noticed the deep connection between Lewis’s axioms for *S*4 and Kuratowski’s axioms for a topological interior operator. They interpreted ‘D’ spatially, as picking out the interior of a region of topological space (more on this below). Finally, Prior interpreted modal languages temporally, and took ‘D’ and ‘3’ to symbolize the temporal sentential operators, ‘Henceforth . . . ’ and ‘At some point in the future . . . ’ (or ‘Until now . . . ’ and ‘At some point in the past . . . ’). These differing viewpoints struck, in some sense, at the heart of the modal logic program: What was modal logic *about*? What modalities did it seek to formalize?

Although Lewis did not concern himself with the problem of giving a formal semantics for modal languages, interest in the subject was quickly growing. Broadly speaking, there were two competing traditions that developed more or less simultaneously: the algebraic tradition, in which modal languages are interpreted in Boolean algebras with operators, and the relational tradition, which culminated in Kripke’s possible world semantics. We focus in this section on the latter, in view of it’s present-day prominence.

An early precursor to Kripke’s possible worlds semantics was proposed by Carnap.[5] According to Carnap, “necessity” was to be interpreted as logical truth, or analyticity (truth in virtue of meaning alone). Infuenced by Leibnitz’s analysis of necessity as that which holds in *all* possible worlds, Carnap introduced the notion of a *state description*. A state description for a propositional language, *L*, is a collection of sentences in which for every propositional variable *P* in *L*, either *P* or $\neg P$ is in the collection, but not both—and nothing else is in the collection. Each state description is a total specifcation of truth for the propositional variables in *L*. We can think of a state description as picking out some possible world, or possible state of affairs, as described by the language *L*. The collection of all state descriptions for *L* is, then, the collection of all possible worlds or states of affair

[5]See (8) and (9).

visible from the point of view of the language, L. In Carnap's semantics, P holds in a given state description if P is a member of the state description; '$\varphi \vee \psi$' holds if either 'φ' holds or 'ψ' holds; '$\neg\varphi$' holds if 'φ' does not hold. Carnap's idea was to analyze necessity, or logical truth, as truth across all state descriptions. Thus, the formula '$\Box\varphi$' *holds in a state description if 'φ' holds in every state description*. [6][7] A number of problems attended Carnap's semantics, some of which Carnap himself recognized. One simple one concerns the failure of standard laws of sub- stitution. In particular, since there is always a state description in which P is true, the formula $\Diamond P$ (or $\neg\Box\neg P$) is true in every state description. Nevertheless, the formula $\Diamond(P \,\&\, \neg P)$ is not true in *any* state description. Taking validity to be truth across all state descriptions, we cannot substitute '$P \,\&\, \neg P$' for 'P' while preserving validity. This violates Lewis's rule of Uniform Substitution, according to which one can substitute arbitrary propositions (sentences) for propositional variables (sentence letters) in valid formulas.[8][9] (Other problems concern the failure of completeness for quantifed $S5$, but we do not go into this here. For a brief discussion, see (4). For a fuller discussion, see (25).)

In the late 1950's, Kripke came up with an idea for a formal semantics for modal logic that effectively did away with this problem. (Kripke was not responding to Carnap—he arrived at this early work independently, while still in high school!) His idea was to interpret propositional modal logic in *partial truth tables*—or truth tables, in which some of the rows are deleted. Each row of the truth

[6] Although Carnap's semantics was developed for frst-order modal languages with the modal operators, '$\Box$' and '$\Diamond$,' we present only the simpler propositional case.

[7] More formally, in Carnap's semantics, we get the following recursive defnition of satisfaction. Let M be a state description, and φ a formula in a propositional modal language, L. Then:

1. $M \models P$ iff $P \in M$ (for any propositional variable, P).
2. $M \models \neg\varphi$ iff it's not the case that $M \models \varphi$.
3. $M \models \varphi \vee \psi$ iff either $M \models \varphi$ or $M \models \psi$.
4. $M \models \Box\varphi$ iff $M' \models \varphi$ for every state description M' in the language L.

The important clause is the modal one. 'Necessarily φ' holds in a state description, M, just in case 'φ' holds in every state description. One important consequence of this defnition of satisfaction is that if the sentence 'Necessarily φ' is true in *one* state description, then it is true in *every* state description. Necessity is not world-relative, as we might say.

[8] The SEP entry, "Modern Origins of Modal Logic," points out that Carnap nevertheless proved completeness of propositional $S5$ for his semantics, but that the proof employs Quine's schematic notion of validity, according to which "a logical truth... is defnable as a sentence from which we get only truths when we substitute sentences for its simple sentences." (32, p. 50)

[9] Somewhat more generally, the problem with Carnap's semantics was that if the sentence 'Necessarily φ' is true in *one* state description (or possible world), then it is true in *every* state description. As a consequence, we cannot have two models of the modal language, on this semantics, in which 'Necessarily φ' is true in one but not the other.

table for a given sentence, 'φ,' is an assignment of truth values to the propositional variables occurring in 'φ'. Again, we can think of these rows as possible worlds in some attenuated sense. In Kripke's early conception, a model for a formula 'φ' in the standard propositional modal language is a pair $\langle G, K\rangle$, where K is some collection of truth assignments for the propositional variables occurring in 'φ,' and G is a member of K. (Thus, a model is a partial truth table in which one row is highlighted.) Each truth assignment in K assigns a truth value to every subformula occurring in 'φ' according to the usual recursive clauses for Boolean connectives, as well as the following rule for the 'D'-modality:

$$\mathrm{D}\psi \text{ is true just in case } \psi \text{ is true in every member of } K$$

Thus, to say that 'φ' is true in a model is to say that 'φ' is true throughout all truth assignments in that model. In this semantics, we say that a formula is *valid* if it is true in every such model.

Notice that under these rules, neither '$3P$' nor '$3(P \vee \neg P)$' is valid! Indeed, if we select only rows of the truth table where 'P' is false, then in this model, '$3P \equiv \neg\mathrm{D}\neg P$' is false. So '$3P$' is not valid. More generally, depending on K—or our selection of rows of the truth table—'$3P$' is true in some models and false in others (and the same for '$\mathrm{D}P$,' or boxed formulas generally). The ability to restrict the collection of possible worlds, K, that matter for the truth of modal formulas is what allows us to do away with the problems faced by Carnap. Kripke showed that the partial truth table semantics is sound and complete for Lewis's $S5$—the strongest of the axiomatic systems introduced in Lewis and Langford (1932).

But what about weaker propositional logics? Consider, for example, the formula '$P \supset \mathrm{D}3P$.' This formula is not a theorem of $S4$, and so should not come out valid in any (complete) semantics for that logic. But the formula is satisfed in every partial truth table. Indeed, if there is some row of the truth table in which P is true, then $3P$ is true in *every* row, and so $\mathrm{D}3P$ is true in every row as well. If, on the other hand, there is no row where P is true, then the formula comes out true in every row in virtue of the fact that the antecedent is false. The simple partial truth tables semantics, while suitable for $S5$ (where this formula *is* a theorem), did not suggest a semantics for the full range of propositional modal logics. In order to give a proper semantics for these systems, a full-blown relational structure had to be developed. (Notice that such structure was implicit—in hindsight—in the simple partial truth tables. If we think of each row in a truth table as a possible world, then a partial truth table consists of some collection of possible worlds, each of which is related to every other.) Such structures had been considered in some form by Hintikka, Kanger, and Prior but it wasn't until Kripke's work in the early 1960's

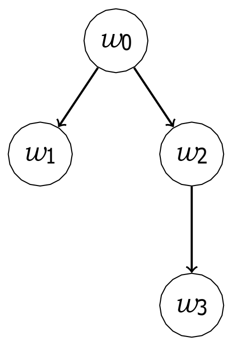

Figure 1: A Kripke frame, $F = \langle w_0, W, R\rangle$, where $W = \{w_0, w_1, w_2, w_3\}$ and $R = \{\langle w_0, w_1\rangle, \langle w_0, w_2\rangle, \langle w_2, w_3\rangle\}$.

that a fully fexible and workable version was articulated.[10]

1.3 Kripke semantics

By now, several of the ideas that appear in the mature version of Kripke semantics [Kripke, 1963] are familiar. The semantics interprets modal formulas in *relational structures* (or frames), which consist of some set of possible worlds, together with a binary 'accessibility' relation on worlds. Pictorially, we can think of a Kripke frame as a graph consisting of some collection of nodes together with arrows pointing from some nodes to others. (See Figure 1.) To say that '$\Box\varphi$' is true at a particular world, w, is to say that 'φ' is true throughout the worlds that are accessible from w. More informally: It is to say that *from the point of view of w, φ is true as far as the eye can see*. (Similarly, to say that '$\Diamond\varphi$' is true at w is to say that 'φ' is true at some possible world accessible from w.)

Formally, a *Kripke frame* is a triple $F = \langle w_0, W, R\rangle$ where W is a set of possible worlds, w_0 is a member of W (the *actual* world), and R is a binary relation on worlds. A *Kripke model* is a pair $\langle F, V\rangle$, where F is a frame, and $V : W \times P \to \{\top, \bot\}$ is a *valuation function*, assigning to each world and propositional variable a truth value (the truth value of that proposition in the given world). We extend the valuation function to the set of all formulas in the language in the way one would expect. In words, '$\varphi \vee \psi$' is true at a world w just in case 'φ' is true at w *or* 'ψ' is true at w; '$\varphi \wedge \psi$' is true at w just in case 'φ' is true at w *and* 'ψ' is true at w; and '$\neg\varphi$' is true at w just in case φ is *not* true at w. But what about the modal symbol, '$\Box$'? The formula '$\Box\varphi$' is true at w just in case 'φ' is true at each world w' such

[10] For a very thorough account of this history, see (13).

that wRw^0. More colloquially, '$\Box\varphi$' is true at a world w if 'φ' is true at all worlds accessible from w. Note that it is the binary accessibility relation that allows us to interpret modalities in Kripke semantics.'[11]

Here we see for the frst time the full-fedged relational framework. Instead of each possible world being related to (or accessible from) every other, we have, as Kripke puts it, a notion of one world being possible *relative to another*.

> We read "H_1RH_2" as H_2 is "possible relative to H_1," "possible in H_1" or "related to H_1"; that is to say, every proposition true in H_2 is to be possible in H_1. Thus the "absolute" notion of possible world in [1959a] (where every world was possible relative to every other) gives way to relative notion, of one world being possible relative to another ... In accordance with this modifed view of "possible worlds" we evaluate a formula A as necessary in a world H_1 if it is true in every world possible relative to H_1. . . . Dually, A is possible in H_1 iff there exists H_2, possible relative to H_1, in which A is true. (20, p. 70) quoted in (13)

The relational structure in Kripke semantics gives us great fexibility. To see, for example, how in this semantics we can refute the formula '$P \supset \Box\Diamond P$,' consider a model consisting of two worlds, w_1 and w_2, where w_1 points to w_2, and P is true at w_1 but not w_2. (See Figure 2.) Here w_2 does not point to any world where 'P' is true, so '$\Diamond P$' is false at w_2. Since w_1 points to w_2, '$\Box\Diamond P$' is false at w_1. It is the relational framework—in particular, the fact that not every world is related to every other—that allows us to fnd a refuting model for this formula in Kripke semantics.[12]

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[11]More formally, we extend the valuation function, V, according to the following recursive clauses:

1. $V(w, \varphi \vee \psi) = \top$ iff $V(w, \varphi) = \top$ or $V(w, \psi) = \top$;
2. $V(w, \neg\varphi) = \top$ iff $V(w, \varphi) = \bot$;
3. $V(w, \Box\varphi) = \top$ iff $V(w^0, \varphi) = \top$ for each $w^0 \in W$ such that wRw^0.

A slightly more complex version of the semantics for predicate modal languages was presented in (20), but in keeping with the focus here on propositional modal logics, we skip over this material.

[12]As is well-known, simple conditions on the accessibility relation correspond to various special axioms of Lewis and Langford's axiomatic systems. For example, if we require that the accessibility relation on worlds is *refexive* (i.e., every world points to itself), we validate the axiom of the system, T: '$P \to \Diamond P$.' Why? If P is true at a given world, w, then since R is refexive, w points to itself. So w points to some world where P is true. This means that the formula '$P \to \Diamond P$' is satisfed in every model defned over a refexive Kripke frame. Moreover, if a frame is non-refexive, then the formula can be refuted in that frame. Consider a world, w_1, which does not point to itself. Let P be true at w_1 and false everywhere else. (See Figure 3.) Here P is false at every world to which w_1 points. So $\Diamond P$ is false at w_1, and we have refuted '$P \to \Diamond P$.' The example shows that Axiom T corresponds

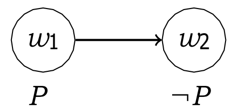

Figure 2: A refuting model in Kripke semantics for the formula '$P \to$ D3P.'

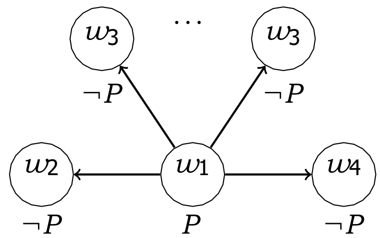

Figure 3: A refuting model for the formula '$P \to 3P$' in an arbitrary, non-refexive Kripke frame. The Kripke frame is non-refexive at the world w_1, which is where we falsify '$P \to 3P$.'

1.4 Space and topological semantics

Relational structures provide a natural setting for interpreting modal languages, but let us now shift gears. We said above that some two decades before Kripke introduced Kripke frames, Tarski noticed a surprising connection between the axioms of Lewis's $S4$, and the axioms used to describe topological space. His work led to what is now called the *topological semantics* for modal logic. Here modalities are interpreted not via a binary accessibility relation between worlds, but via the topological structure of space. To understand the semantics, we need to say something about what a *topology*, or *topological space* is.

1.4.1 A mathematical view of space

In our ordinary lives, we have a number of well-entrenched views about space and spatial properties. Any two distinct points bear a precise distance relation to one another. A sequence of points that converges, converges to a single point. No two points are infnitely far away. And so on. From a mathematical point of view, these features of space are not universal. When we think about space mathematically, we think in more general terms: there are many different *kinds* of space, with different spatial properties. For example, not all spaces come with a notion of 'distance.'

---

to the class of refexive Kripke frames. Similar arguments show that other axioms correspond to the class of transitive frames, symmetric frames, and frames in which R is an equivalence relation.

In some spaces, it is impossible to say that one point stands three units away from another. Indeed, spaces that *do* allow for a notion of distance are rather special: we call them metric spaces, or spaces that have a metric (read: distance) function defned on them. What, then, is space in the fully general, mathematical sense that we are after?

A space, as we think of it here, is just *a collection of points that are glued together in a certain way.*

There are two ways to understand this. The frst involves the notion of a *neighborhood*, or as mathematicians say, *open* set.[13] Think of the city of London. That city is made up of a very large number of different points on the earth that lie inside of its municipal boundaries. These points lie at various distances from one another: the Big Ben is (let us suppose) one mile from the Tate Modern, which is itself another half mile from the London Eye. But quite apart from specifc distances, there are also neighborhoods in London: Hampstead, Notting Hill, Chelsea, and so on. Some of these neighborhoods overlap; others are disjoint. Imagine throwing out all information about the relative distances between individual points in the city. London, as you view it now, is a collection of points linked together by a system of neighborhoods. The information about neighborhoods furnishes some sense of how points in this space are related to one another spatially. When we speak of space mathematically, in a completely general way, we view it in this way: as a collection of points together with a system of neighborhoods, or open sets.

These open sets, or neighborhoods, must satisfy certain conditions if they are to defne a topology on the underlying set of points. In words these conditions state that the entire space and the empty set are open; the intersection of any two open sets is open; and fnally, the union of any collection of open sets is open. More formally, a topological space is a pair, $\langle X, \mathsf{T} \rangle$, where X is a set (of 'points'), and T is a collection of subsets of X that satisfes the following conditions:

1. $X \in \mathsf{T}$, $\emptyset \in \mathsf{T}$;
2. If $S_1, S_2 \in \mathsf{T}$, then $S_1 \cap S_2 \in \mathsf{T}$;
3. If $\{S_i \mid i \in I\} \subseteq \mathsf{T}$, then $\bigcup_{i \in I} S_i \in \mathsf{T}$.

We call the sets in T *open*. Any collection of subsets of X that satisfes these conditions defnes a *topology* on X. Again, space according to this defnition consists of a collection of points together with a system of open sets, or neighborhoods.

A second, less familiar way to think about space is as a set of points together with an *interior* operator. This operator identifes, for any subset of points, what the

[13] Although it is standard to use the expression 'neighborhood of x' to mean any set containing an open set containing x, we use the term 'neighborhood' to mean, simply, open set.

interior of that subset is. Think of the interior of a region as the region minus any *boundary* points. For example, if we start out with a potato-shaped region of three-dimensional Euclidean space, the skin of the potato is the boundary, and the interior is the white, feshy stuff inside. Or, starting with a disc in two-dimensional space, the circumference of the disc is the boundary, and everything else is the interior. There may be regions of space that have *no* interior. For example, the region of the real plane consisting of a point at (0,0), another point at (1,0), another point at (2,0), and so on. This region is *all* boundary. Or, there may be regions of space that have no boundary. Consider, for example, the open disc in two-dimensional space—the disc without any of the points along its circumference. This region is *all* interior. Information about the interior of each region of space again gives us some conception of how points in the space ft together spatially. On this way of viewing things, we think of space as a collection of points, together with information about what the interior of each region, or subset of points, is.

Again, the interior operator must satisfy certain conditions in order to count as an interior. In words these conditions state that the interior of any region is a subset of that region; the interior of the whole space is the space itself; the interior of the intersection of two regions is the intersection of their interiors; and fnally, the interior of the interior of a region is just the interior of that region. (Iterating interiors gives us nothing new.) More formally, let X be a set of points, and let A and B be arbitrary subsets of X. Then an interior operator, *I*, on *X* must satisfy:

(1) $I X = X$.

(2) $I A \subseteq A$.

(3) $I(A \cap B) = I A \cap I B$.

(4) $II A = I A$.

On this conception of space, a topology consists of a set of points, *X*, together with an interior operator, *I*, on *X*. Again, any operator that satisfes these conditions defnes a topology on *X*. (Thus, there may be many *different* topologies on any given set of points.)

Although these two ways of viewing space may seem quite different, from a mathematical point of view they are interchangeable. Starting from a collection of open subsets of *X*, we can defne the interior of any set $S \subseteq X$ to be the union of all open sets contained in *S*:

$$\text{Interior}(S) = \bigcup \{O \text{ open} \mid O \subseteq S\}$$

Or, starting from an interior operator on X, we can defne an open set as a set that is equal to its own interior:

$$S \text{ is } open \text{ if and only if } Interior\,(S) = S$$

The technicalities here are, for the moment, not essential. The point is just that information about spatial structure is encoded in the collection of open sets, or alternatively, the topological interior operator.

1.4.2 Topological semantics

But what does any of this have to do with modal logic?

In the late 1930's, McKinsey and Tarski were studying what they thought of as the 'algebra' of topology. A topological space can be represented as the Boolean algebra of all subsets of the space. Here the interior operator is conceived of as an operator on the algebra itself, taking elements of the algebra (subsets of points) to other elements of the algebra.[14] Thus, a topological space is represented as a Boolean algebra with an operator. Viewed in this way, Kuratowski's axioms are really just algebraic equations. They tell us that the interior of the top element in the algebra is equal to the top element; the interior of any element is less than or equal to that element; the interior of the meet of two elements is equal to the meet of the interiors; and fnally the interior of the interior of any element is equal to the interior of that element. More formally, the algebraic analogs of (1) - (4) are:

(1*) $\mathrm{I}\,1 = 1$.

(2*) $\mathrm{I}\,a \leq a$.

(3*) $\mathrm{I}(a \wedge b) = \mathrm{I}\,a \wedge \mathrm{I}\,b$.

(4*) $\mathrm{II}\,a = \mathrm{I}\,a$.

where '1' denotes the top element of the algebra.

But here now was a curious thing. Substituting '$\Box$' for 'I' in these equations,[15], and rearranging things a bit, what we get is just the axioms for the modal logic $S4$!

[14]Meets, joins and complements in the algebra are, respectively, set-theoretic intersections, unions and complements.

[15]And, of course, making the appropriate substitutions for Boolean connectives — in particular, replacing '$\subseteq$' with '$\rightarrow$,' '$\cap$' with '&,' and '=' with '$\leftrightarrow$.'

Or, conversely, substituting 'I' for 'D' in the axioms for $S4$, what we get is the algebraic version of Kuratowski's axioms for a topological interior.[16] In other words, a set of axioms introduced by Lewis and Langford to formalize the ordinary notions of possibility and necessity were the very same axioms (under this translation) that describe topological space, or space as it is understood mathematically!

This discovery must have been quite surprising. A friend of mine, D. Sarenac, likes to imagine the following scenario. It is sometime in the early 1930's and C. I. Lewis and C. H. Langford are puzzling over what exactly the new axioms for modal logic should be. Langford is tending to the fre; Lewis is sitting in an armchair nearby, pen and paper in hand. The two men are engaged in the following conversation:

Lewis: So, Langford, about those axioms for our new system of 'necessity' and 'possibility'. . .

Langford: Yes?

Lewis: Well, I was wondering. Suppose that 'Necessarily P' is the case. Does it follow that P is the case?

Langford: Yes, sir, I believe it does. If P is *necessarily* true, then P must, at the very least, be *true*, right?

Lewis: Okay, I'm with you there, Langford. But how about this. Suppose 'Nec-

[16]The modal logic $S4$ in the language L consists of some complete axiomatization of classical propositional logic, PL, some complete axiomatization of the minimal normal modal logic, K, say the axiom:

$$K: \mathrm{D}(\varphi \to \psi) \to (\mathrm{D}\varphi \to \mathrm{D}\psi)$$

and the rule:

$$N: `\ \varphi \Rightarrow\ `\ \mathrm{D}\varphi$$

together with the special $S4$ axioms:

$$T: \mathrm{D}\varphi \to \varphi$$

$$4: \mathrm{D}\varphi \to \mathrm{D}\mathrm{D}\varphi$$

With a bit of work, K together with N yield: $\mathrm{D}(\varphi\ \psi) \to (\mathrm{D}\varphi\ \mathrm{D}\psi)$. This states that the intersection of the interiors of two regions is equal to the interior of the intersection of those regions. N states that the interior of the entire space is the space itself. T states that the interior of a region is a subset of that region. Finally, 4 together with T states that the interior of the interior of a region is just the interior of that region. The connection to Kuratowski's axiomatization of the interior operator should now be clear.

essarily P' is the case. Does it follow that 'Necessarily, Necessarily P' is the case?

Langford: That's a tough one, sir...

Reasoning in this way, the two men arrive at a system of axioms and rules of inference that they think captures the English-language 'necessity' and 'possibility' modalities.[17] How extraordinary that these axioms should coincide perfectly with Kuratowski's axioms for topological space!

In the topological semantics, a model consists of a topological space together with a valuation function. Here formulas are true or false not at a *possible world*, but at a *point* in a given topological space. The Boolean connectives are interpreted in just the way you would imagine: a disjunction is true throughout the *union* of the set of points where each disjunct is true; a conjunction is true throughout the *intersection* of the set of points where each conjunct is true; and a negation is true throughout the *complement* of the set of points where the negated formula is true. The important clause, as always, is the modal one, and this is where the topological semantics gets its name. The formula '$\mathrm{D}\varphi$' is true throughout the *interior* of the set of points where 'φ' is true. We say that a formula is *satisfed* by the model if it is true throughout the entire space, and is *valid* in the space if it is satisfed in every model defned over the space.[18]

What these defnitions tell us is that *each topological space picks out, semantically speaking, some set of modal formulas*—namely, the set of formulas that are valid in that space. In other words, to each topological space is associated some collection of sentences in the given propositional modal language. But now we can ask some fundamental questions: Do different topological spaces pick out different sets of formulas? Moreover, is the set of formulas picked out by a given topological space axiomatizable? Does it coincide with the theorems of any known axiomatic

[17] In fact, of the fve axiomatic systems for modal logic that Lewis and Langford proposed, they were said to favor the system $S2$ as a formalization of the English language 'necessity' operator.

[18] More formally, in the topological semantics a model consists of a pair, $\langle X, V\rangle$, where X is a topological space and $V : \mathsf{P} \to \ (X)$ is a valuation function that assigns to each propositional variable P some subset of the space X. We extend the valuation function, V, to the set of all formulas by the following recursive clauses:

$$V(\varphi \vee \psi) = V(\varphi) \cup V(\psi)$$
$$V(\neg\varphi) = X \setminus V(\varphi)$$
$$V(\mathrm{D}\varphi) = \mathrm{Int}(V(\varphi))$$

We say that a formula, 'φ,' is *satisfed* in the model if $V(\varphi) = X$. Finally, 'φ' is *valid* in X if 'φ' is satisfed in every model defned over X.

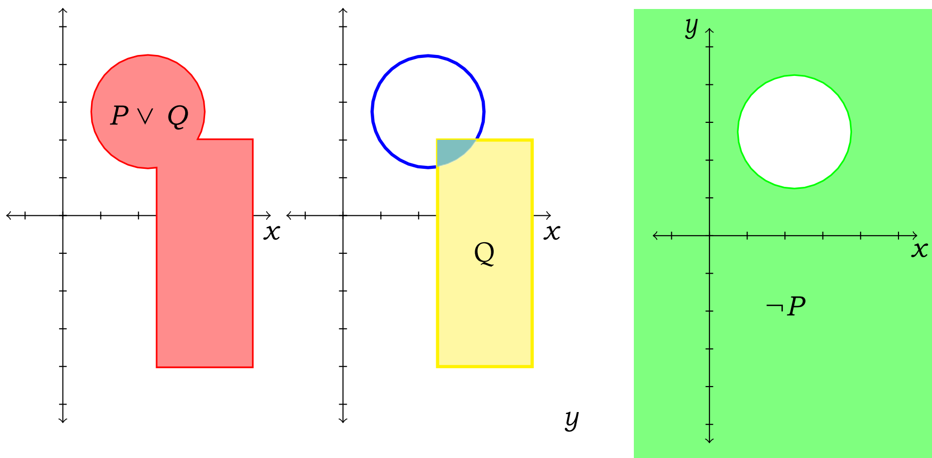

Figure 4: A topological model in the real plane, $\mathbb{R}^2$. Here P is true throughout the disc; Q is true throughout the rectangular region of space.

system? In particular, does it coincide with the theorems of $S4$? This last question can be broken down into two more specifc ones: Is it the case that every theorem of $S4$ is valid in the given space? And: Is it the case that every formula valid in the space is a theorem of $S4$? Respectively: Is $S4$ *sound* and *complete* for the topological space?

Soundness, you will have noticed, is had for free. Indeed, this is the real content of the connection between the $S4$ axioms and Kuratowski's axioms for topology. The axioms of $S4$, interpreted topologically, just restate the conditions that a topological interior must satisfy in order to count as an interior operator at all. So of course they are valid. But what about completeness? Here things are much more complex. Completeness of $S4$ for a given topological space, X, is the claim that every validity in X is provable in $S4$:

$$\models_X \varphi \quad \Rightarrow \quad \vdash_{S4} \varphi$$

It is helpful to restate this claim in an equivalent way, by taking the contrapositive. Thus, completeness says that if φ is *not* a theorem of $S4$, then φ is *not* valid in X. In symbols,

$$\not\vdash_{S4} \varphi \quad \Rightarrow \quad \models_{X} \varphi$$

Putting things this way allows us to see that completeness is really a claim about the fexibility of a given topological space—the availability, in that space, of a broad enough class of refuting models.

In 1944, Tarski and McKinsey proved a very strong completeness result that is sometimes called the Tarski Theorem, and which is in some ways the culmination of their work on the topological semantics. The result is that $S4$ is complete for any dense-in-itself metric space. (A *metric* space is a space in which we can defne a distance function; a *dense-in-itself* space is a space where every point is the limit of other points in the space.) Dense-in-themselves metric spaces include the most familiar and widely studied topological spaces—for example, the real line (indeed, any fnite dimensional Euclidean space), the rationals, Cantor space, and so on. If we think of Euclidean space as *our* space, then the Tarski theorem says that $S4$ is the logic of space as we know it.[19]

1.5 Measure and probabilistic semantics

Tarski's work is part of an algebraic tradition in modal semantics, in which formulas are interpreted not in relational structures, but in Boolean algebras with operators. The idea here was that just as classical propositional logic is interpreted, at the most general level, in Boolean algebras, so too propositional modal logic should be interpreted in Boolean algebras with operators that interpret the new modal symbols. What kind of operators? The modal axioms of an existing axiomatic system dictate what is needed. We saw that in the case of $S4$, the appropriate operator was one that satisfes Kuratowski's four axioms—or the algebraic version of those axioms. But once we've put things in this general algebraic way, it's clear that we need not restrict our attention to Boolean algebras that arise, in the way described, from pointed topological spaces. Indeed, we can interpret $S4$ in *any* Boolean algebra together with an interior operator that satisfes (1*) - (4*).[20]

What other algebras are of interest? Here we should recall the second of the perspectives on the topological semantics mentioned at the beginning of this chapter. Beginning with an interest in existing mathematical structures—namely, topologies—we take interest in the topological semantics because it allows us to describe these structures using modal languages. What we are interested in is, to reiterate, such questions as: What logics do such structures give rise to? What is the expressive power of modal languages vis-à-vis these mathematical objects? To

[19] The Tarski theorem can be seen in both a positive and negative light. On the positive side, it tells us that any dense-in-itself metric space has the resources to refute all non-theorems of $S4$. Viewed in this way, the result is, again, a statement about the availability of counter-models in the given topological space. Fixing a topological space, we can ask of interesting modal formulas, what do such refuting models look like? What is their *geometry*, say, on the real line? On the negative side, the result groups together many different spaces that have quite different features. To say that $S4$ is the logic of any dense-in-itself metric space is to say that as far as the sentences of the basic modal language go, *we cannot tell these spaces apart.* This says something important about the expressive power of the basic modal language, interpreted topologically.

[20] Of course, there is no guarantee that this will yield a complete semantics, only a sound one.

what extent can such languages describe and discriminate between different kinds of topological spaces? This point of view is quite natural. Topologies are fundamental objects in mathematics, and the fact that we can talk about them in a formal modal language is entirely non-trivial. Tarski's completeness results show that $S4$ characterizes any dense-in-itself metric space—and in particular, Euclidean space of any fnite dimension.

But Euclidean space has, in addition to topological structure, *measure* structure. Different subsets of the reals have different size or measure. This measure structure is quite distinct from topological structure. Sets that appear large from a topological point of view may be small, or insignifcant from a measure-theoretic point of view. (Take, for example, the rationals, which are dense in the reals, but have measure zero.) What if we could interpret modal languages in a Boolean algebra that encoded not simply the topological structure of Euclidean space, but it's measure structure?

1.5.1 Measure

To get a feel for measure, consider the following simple game. You have, in front of you, a ruler which is exactly one meter in length. The left end of the ruler is marked by a zero, and the right end is marked by a 1; points in between are marked by their distance in meters from the left endpoint. Your opponent chooses a region of the ruler, by specifying any set of points that she likes, and a dart thrower prepares to throw one hundred darts at the ruler in sequence. Your job is to guess how many of those darts will land within the region of the ruler selected by your opponent. The closer your prediction is to the actual outcome, the more points you make in the game. Assuming that the darts land on the ruler in a more or less random fashion, what should your strategy be?

Here we can make a number of simple observations. If your opponent selects any *interval*, then the probability that the dart lands in that interval is equal to the length of that interval. So, for example, if your friend selects the interval $[\frac{1}{4}, \frac{3}{4}]$, the probability that a random dart lands in the selected region is $\frac{1}{2}$. Likewise, if your friend selects some fnite union of disjoint intervals, say $[\frac{1}{4}, \frac{1}{2}] \cup [\frac{3}{4}, \frac{7}{8}]$, the probability that the dart lands in the selected region is equal to the sum of the lengths of the intervals, or $\frac{3}{8}$. But what if the region selected is more complex? What if your opponent selects, *e.g.*, the collection of all rational points in the interval? Or all irrational points? Is there a way of saying, for *any* region of the ruler, what the probability of hitting that region is? Here we run into some practical obstacles. For example, although your opponent would have no trouble *naming* the region of the interval consisting of all rational points, there would be no way to determine—indeed, no fact of the matter—whether the dart landed in that region or not (given that the dart has non-zero thickness). Likewise for sets like the irrationals, the Cantor set, and so on. More troubling still, assuming the Axiom of Choice, there are regions of the ruler that cannot even be named. This is because such regions are not constructible—they cannot be picked out explicitly. Nevertheless, if we accept the Axiom of Choice, such regions do exist. Is there a well-defned probability that the dart lands in one of these regions?[21] What we are after here is a notion of *measure*. We would like to know what proportion of the interval or the ruler is taken up by any given set, so that we can say what the probability is of a random dart landing in that set.

In 1901, H. Lebesgue defned what is now the standard measure on the real line. He showed that while there is no way to defne a 'nice' measure on every subset of the reals which extends the notion of length for intervals, we can defne such a measure on a very large and important class of subsets (the Borel subsets, or more generally, Lebesgue-measurable subsets). Without going into the mathematical details, we can describe this measure by saying that it (1) extends the notion of length for intervals, (2) is translation-invariant, and (3) is countably additive. In other words, the measure of any interval is equal to its length; "pushing" a mea- surable set up or down the real line does not change its measure; and the measure of any countable union of disjoint sets is the sum of the measure of the individual sets.[22] When we restrict Lebesgue measure to the interval [0, 1], as we've been doing, this function captures the familiar notion of probability. The *measure* of a given region of the interval is just the *probability* that a dart hitting the interval at random lands in that region (leaving aside practicalities having to do with the thickness of the dart). It is important to note that there are many subsets of the real interval [0,1], which are non-empty but nevertheless have measure equal to *zero*. The simplest example is a singleton set $\{a\}$, where a is any point in the interval [0,1] (see Note 22). In the game we described, the probability of hitting any one of these sets with a dart is precisely zero. *This does not mean that this event cannot occur.* Events which have probability, or measure zero, are not *impossible*; it is simply that no fnite number, however small, can capture the likelihood of their

[21] It is well-known that one needs the Axiom of Choice to prove the existence of non-measurable subsets of the reals. Thus someone who denied the axiom could insist that all subsets that *really* exist are measurable. Here one is reminded of Bill Clinton's famous line: "It depends on what the meaning of the word 'is' is."

[22] This already allows us to answer some of the questions posed above. The measure, for example, of the set of rational points is precisely zero. Why? The measure of any individual point is the same as the measure of any other (by translation-invariance). Moreover, the measure of an individual point is zero, because if it were non-zero, any countably infnite collection of points in the real interval [0, 1] would have infnitely large measure (by countable additivity)! But if each point has measure zero, then any countable set has measure zero as well (again, by countable additivity). Since the rationals are countable, the measure of the rationals is zero.

occurrence.

From a measure-theoretic point of view, measure-zero subsets of the reals are insignifcant. Taking this thought seriously, what if we were to literally *ignore* the existence of such sets? What if, in other words, we were to identify any two subsets of the reals that differ from one another by a set of measure zero? Imagine, if you will, that you have blurry glasses, and that these glasses do not allow you to distinguish between such sets.[23] Seen through these glasses, the real interval, $[0, 1]$, consists, for you, of some collection of blurry regions, each of which has a precise measure (namely, the measure of any one of the sets which make up that region). Formally, these regions make up a Boolean algebra: the algebra of all measurable subsets of the real interval, *modulo sets of measure zero*. This is a measure algebra — or Boolean algebra, in which each element has a measure between 0 and 1.[24] We call it the *Lebesgue measure algebra*.

1.5.2 Probabilistic semantics

The Lebesque measure algebra encodes information about the measure structure of the real line. Just as we used Boolean algebras generated by topological spaces to get a topological semantics for modal logic, Scott's idea was to use measure algebras to get a probabilistic semantics for modal logic. Formally, a probabilistic model in the basic propositional modal language is a pair, $\langle M, V \rangle$, where M is the Lebesgue measure algebra, and $V : P \to M$ is a valuation function that assigns to each propositional variable some element of the algebra M. We would like to extend the valuation function to all formulas in the language by a recursive truth defnition. For Boolean connectives, the defnitions are straightforward:

$$V(\varphi \vee \psi) = V(\varphi) \vee V(\psi)$$

$$V(\neg\varphi) = -V(\varphi)$$

but how to interpret the D-modality? Here of course, we must construct an interior operator on the algebra M but we've said nothing at all about how to do this. Indeed, how can we be sure that there *is* non-trivial interior operator on this algebra?

The key, again, is to consider the topological structure of the reals from an algebraic point of view. Just as there are open subsets of real numbers, so too we defne open elements of the Lebesgue measure algebra. We say an element of the algebra is *open* if it has some representative which is an open subset of the real

[23] The metaphor of seeing through blurry glasses is due to F. Artzenius. See (1).

[24] More formally, a measure algebra is a Boolean σ-algebra with a positive, normalized measure. See Defnition 3.3.4 below.

interval, [0, 1].[25] Thus, for example, the element of the algebra corresponding to any interval, or any fnite disjoint union of intervals, is open. (Take as your representative set the interval without its endpoints.) But recall the interchangeability of the two ways of defning topological structure described in Section 4.1. Once we have open sets we have an interior operator, and vice versa. In topology, we defne the interior of a set, A, as the union of open sets contained in A:

$$\bigcup \{C \text{ open} \mid C \subseteq A\}$$

The algebraic analog of this topological defnition is not hard to fnd. Indeed, we defne the interior of an element, a, in the Lebesgue measure algebra as the supremum of all open elements dominated by a:

$$\bigvee \{c \text{ open} \mid c \leq a\}$$

(One of the deep lessons of Tarski's work in this area is that topological defnitions are essentially algebraic, whether we focus on those conditions placed on open sets or on an interior operator.) Completing the recursive defnition of truth for the probabilistic semantics, we have:

$$V(\mathrm{D}\varphi) = I(V(\varphi))$$

In the probabilistic semantics, we interpret the basic propositional modal language in the modally-expanded Lebesgue measure algebra—or algebra together with interior operator. Each formula is assigned to some element of the algebra, and thus acquires the probability—or measure—value associated with that element. We say that a formula 'φ' is satisfed in a probabilistic model if the value of that formula is the top element of the algebra (*i.e.*, $V(\varphi) = 1$.) Equivalently, 'φ' is satisfed if the probability of 'φ' is 1.

The modally-expanded Lebesgue measure algebra encodes information about both the topological and measure structure of the real line. Other topological spaces and measures give rise to different measure algebras. But now we can ask all of the familiar questions that we asked about the topological semantics in this new setting. Do different measure algebras give rise to different sets of validities? Is the set of validities of the Lebesgue measure algebra axiomatizable? If so, does it correspond to any known axiomatic system? In particular, does it correspond to the theorems of $S4$? In other words, is $S4$ sound and complete for the probabilistic semantics? At the time I began work on this project, these questions had not yet

[25] Note that the collection of open elements, so defned, satisfes the algebraic analog of the conditions on open sets. In particular, the top and bottom elements of the algebra are open, the meet of any two open elements is open, and the join of an arbitrary collection of open elements in open.

been answered. Indeed, much of the work that occupies the chapters ahead is devoted to settling some of the most pressing among them. The third and fourth chapters, in particular, show that $S4$ is complete for the Lebesgue measure algebra; that the probabilistic semantics can be extended to more complex, dynamic modal languages, and fnally, that we get nice completeness results here too. Reading between the lines, the reader will also discover what I take to be something of an analog to the Tarski Theorem in the measure-theoretic setting.

It is important, before getting lost in the mathematics, to reiterate two different perspectives from which we might approach the probabilistic semantics developed in these pages. On the one hand, one might take interest in the fact that we have here a new semantics for existing axiomatic systems—one, moreover, with probabilistic features that set it apart from other well-known semantics. This point of view may be attractive to those philosophers dealing in the many formal areas of philosophy which exploit probability theory for philosophical purposes. Indeed, the new semantics provides a very general and fexible framework for attaching probability values to formulas in rich, modal languages in a systematic way. It is not implausible to think that this could be of use in such areas as Bayesian epistemology and rational choice theory, where we model agents as having precise *credences* in propositions, and not just full-fedged *beliefs*. Other applications may be found in philosophy of language—in particular, where it comes to understanding the various components of meaning in natural language. In an early paper addressing such issues, H. Field argues that Popper's probabilistic semantics for classical logic can be put to that use. Indeed, Field understands agents as attaching conditional probabilities to formulas in a classical language, and argues that the conceptual role component of meaning should be understood in terms of these probabilities. According to Field, two propositions, P and Q, have the same conceptual role for an agent, S, just in case for any proposition C,

$$Prob_S\,(P|C) = Prob_S\,(Q|C)$$

where $Prob_S$ is the conditional probability function representing S's beliefs. Field shows that Popper's probabilistic semantics for classical propositional logic can, with some effort, be extended to predicate logic. Although there are signifcant differences between Popper's semantics for classical logic and the probabilistic semantics presented here, the latter does give us the tools to interpret not just classical languages, but rich modal languages probabilistically. The greater expressive power of these languages, as well as the ease with which the present framework can be exported to predicate and multi-modal settings, may prove useful to those sympathetic to Field's endeavor.[26]

[26]See (12).

On the other hand, one might approach the new semantics from a more mathematical point of view. Just as topologies are basic objects in mathematics, so too are Borel measures. As we've seen, topologies together with Borel measures give rise to modally-expanded measure algebras. To what extent can modal languages describe, discriminate between, and express the various properties of these measure structures? There is, in my view, no *one* correct way to think about the probabilistic semantics: each of the perspectives announced here has its merits, and will, with luck, yield new ways of developing the work begun here. But before launching into that work, I want to briefy mention one surprising philosophical application. The application is in the feld of metaphysics, or mereology.

1.6 Gunk via the Lebesgue measure algebra

Space as we conceive of it in mathematics and physics consists of dimensionless points. We typically describe not just positions in space, but trajectories, velocities and accelerations in terms of three-dimensional spatial coordinates. Over the years, however, some have sought to deny that points, or point-sized parts, are genuine parts of space or matter. In the words of P. Roeper,

> Points are not parts or elements of space; a point is a location in space. As a consequence, points are not the primary bearers of spatial properties and spatial relations, nor the primary objects of spatial mappings. This role belongs rather to the parts of space.

According to what is sometimes called a 'gunky' picture of space, space consists of regions that can be *arbitrarily* small, but no region is literally dimensionless. Space, on this conception, is not *chunky*—there are no smallest bits of space which cannot be broken up further—but rather *gunky*—each region can be further broken up into smaller regions. We can put this loose picture of space in the form of a more precise mereological thesis:

GUNK (S): Every region (part) of space has a proper subpart.

The thesis as stated is a thesis about physical space. But there is, of course, a parallel thesis about physical matter that could, in principle, be held independently of any view about space:

GUNK (M): Every region (part) of matter has a proper subpart.

It's easy to see that these principles rule out the existence of point-sized bits of space or matter. As points in space are literally dimensionless, they cannot be further broken down into proper subparts.[27]

1.6.1 Motivations

There may be many motivations for adopting a gunky conception of space or matter, some more prosaic, and others reaching deep into phenomena in mathematics and physics.[28] Frank Arntzenius, for example, motivates a gunky conception of time by reference to Zeno's paradox, and argues as follows. Zeno argued that if time consists of instants of zero duration, then an object is always stationary during a single instant of time. So an object is never in motion. But if objects are never in motion, how do they succeed in *moving*?

Aristotle's response to the paradox was to relinquish the idea that there are zero-sized instants of time. Instants of time can be of arbitrarily small duration, he thought, but no single instant has duration strictly equal to zero. Although this move avoids the problem raised by Zeno, one might think that there are other, less costly responses to the paradox. To be in motion, one might say, is just to be in different locations at different instants of time. The fact that at a single point-sized instant of time an object is stationary is not, on this response, a problem for the possibility of motion at all. Stationary objects at single instants is just the stuff of which motion is made.[29] On this response, however, motion (or velocity) is not an *intrinsic* property of the state of an object at a given moment. Indeed, motion as conceived of here is a property that arises from the relationship between the state of an object at one time, and the state of that object at past or future times. If that is the case, it would seem wrong to say that the instrinsic state of an object (or of all objects in the world) at any given instant determines the state of that object at future instants. In short, some form of physical determinism seems to be threatened here. [30] (1)

[27] One could, in principle, affrm one of these two theses about gunk and deny the other. Thus, one could believe for example that while matter is gunky, space is not. Arntzenius and Hawthorne argue against this sort of split position. "If we are to restrict the Difference thesis to material objects, we need some reason for tolerating zero measure differences in the domain of spatiotemporal objects while prohibiting them within the realm of the material. We are not aware of any such reason." (2). In what follows, we focus primarily on the thesis GUNK (S), but the reader who is interested only in a gunky view of matter can make the appropriate substitutions.

[28] These arguments appear in (1), (2) and (35).

[29] There is something bizarre, to my ear, about saying that an object is 'stationary during a single instant.' To be *stationary*, one wants to say, is to be in the same place across some stretch of time.

[30] One might argue that the sort of determinism here threatened is too strong to be plausible. Why, after all, should it be only the *intrinsic* properties of an object that determine its future states? On the other hand, if motion is a product of the relationship between states of the object at one time and

Another motivation for gunk, again discussed in (1), is more mathematical in nature. It is well known that if we admit points in space, then using the Axiom of Choice, we can prove the existence of non-measurable subsets of Euclidean space. The existence of such sets, however, leads to the Banach-Tarski paradox. What Banach and Tarski showed is that we can divide a sphere in three-dimensional Euclidean space into fnitely many parts (fve, to be precise), move these parts around without stretching or deforming them in any way (thus, performing only rigid motions), and end up with a sphere that has twice the volume of the original. This result is quite startling, and depends on the existence of non-measurable sets. (Recall that Lebesgue measure is translation-invariant. Thus some of the parts into which we divide the sphere must be non-measurable.) One response to the paradox is, of course, to give up the Axiom of Choice. Without that axiom we would not be able to prove the existence of non-measurable sets, and so we would not be able to divide the original sphere into the kind of parts needed to get the paradox going. But although the Axiom of Choice was initially greeted with controversy, it is now accepted by practically all practicing mathematicians. Retaining all of pointy mathematics while doing away with the axiom is not a realistic option. Another response would be to deny the existence of point-sized regions of space (or matter). On this response, we can allow that points and the axiom have a role to play in mathematics, but deny that they have a similar role to play in the correct understanding of physical space and matter. In other words, we can preserve points in a purely abstract, mathematical setting, while at the same time staving off the idea that a sphere-shaped region of space or matter could be doubled at no cost.

These motivations do not form anything like a complete list, and even as they stand are quite tenuous.[31] Nevertheless, in the words of F. Arntzenius and J. Hawthorne, "The idea that all physical objects are gunky seems suffciently sweeping, interesting, and plausible that it is worth examining." (2, p. 441)[32]

1.6.2 The approach based on regular closed sets

Suppose then, for the moment, that space and/or matter really are gunky. The question now arises: How should we model space mathematically? What model of space respects standard mereological assumptions together with the gunky picture of space sketched above?

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states of the object at future times, it seems wrong to say that the state of motion of an object at an instant *determines* future states (whether or not motion is an intrinsic property). It is in part the future states, one wants to say, that make up motion to begin with.

[31]For a much more thorough discussion of motivations, see (1).

[32]Here the gunk thesis is defended for physical objects, but as we saw above, Arntzenius and Hawthorne do not think it plausible to adopt a gunky conception of matter together with a pointy conception of space. See (2).

This challenge—as well as the traditional response to it—is most famously associated with A.N. Whitehead. As early as the 1920's, Whitehead was investigating the possibility of doing geometry without points. The idea was, as Biancino and Gerla point out, to conceive of "axiomatic systems in which the concept of a point is defned from primitive terms more easily interpretable in nature." [33] Whitehead took as a topological primitive the notion of two regions being *connected*. Intuitively, a region A is connected to B if A and B overlap, or at least share some boundary point. In 1929, Whitehead showed that his axiomatization of this relation was satisfed in regular closed algebras (defned below), and that one could use such algebras to model pointless geometry. Points, lines and surfaces were constructed as mathematical abstractions on these algebras of solid regions.

Recall that a *regular closed* set is a set that is equal to the closure of its own interior. In symbols:

$$A = Cl(Int(A))$$

where the closure of a set is the set together with its topological boundary. The simplest example of such sets is a closed sphere in n-dimensional Euclidean space. (The interior of the set is the open sphere, and the closure of the interior is the original, closed sphere.) [34] The algebra of regular closed sets can be constructed from the collection of all subsets of a topological space in the following way. Starting with the set of all pointy subsets of a space, we write $A \sim B$ if the closure of the interior of A is equal to the closure of the interior of B. It is not diffcult to see that the relation '$\sim$' so defned, is an equivalence relation. Taking equivalence classes we get a Boolean algebra in which each element of the algebra has exactly one regular closed representative—that set equal to the closure of the interior of any set in the equivalence class.[35]

Modeling gunky space in regular closed algebras brings with it many advantages. The Boolean structure of the algebra satisfes standard mereological assumptions, which we do not repeat here.[36] Moreover, the regular closed algebra

[33] See (6), p. 431.

[34] For an example of a closed set in one-dimensional Euclidean space that is *not* a regular closed set, consider the Cantor set. This set has no interior, so the closure of its interior is empty.

[35] Operations in the algebra of regular closed sets are defned as follows:

$$a \wedge b = Cl\,(Int\,(A \cap B))$$
$$a \vee b = A \cup B$$
$$-a = Cl(X - A)$$

where X is the entire space, A and B are regular closed subsets of X, and a and b are the corresponding elements of the algebra.

[36] In fact, standard mereology does not admit a null element, whereas such an element is present in the algebra of regular closed sets.

that arises from fnite-dimensional Euclidean space (indeed, from any *Hausdorff* topology) is *non-atomic*: every non-zero element in the algebra dominates some other, non-zero element. More formally, for every non-zero element a in the algebra, there is an element b such that $0 < b < a$. In words, every region of space has a proper subregion. This fact about regular closed algebras is precisely what is required by the thesis GUNK (S) given above. But, as J.S. Russell points out, Whiteheadian space is not just mereologically distinctive, but also topologically distinctive. To see this, let us say that a region x is an *interior part* of y if $x \leq y$, and x is not connected to any region disjoint from y (where two regions a and b are disjoint if there is no region that is part of both). Then in Whiteheadian space, *every region has an interior part*. Russell calls space that satisfes this condition 'topologically gunky.' As he argues, "Topological Gunk is a natural extension of Mereological Gunk: not only does every region have a proper part, it has a part which is strictly inside of it."[37]

More recently, however, powerful arguments have been leveled against the idea of interpreting space (or at least actual, physical space) in regular closed algebras. These arguments stem from diffculties associated with defning a reasonable measure on the algebra. Indeed, the gunk theorist would like to be able to talk not just about mereological structure, but also about the *size* of various regions of space. Ideally, he would like to be able to say, of any region of space or matter, what the size of that region is. Moreover, in keeping with the spirit of gunk, many gunk theorists would add that no region of space has size equal to zero. We can put these desiderata concerning size in the form of two additional theses:

SIZE: Every region (part) of space has a precise size.

NO ZERO: No region (part) of space has size equal to zero.

The notion of size in play here is not one of, *e.g.*, cardinality. In talking about the size of a region, we distinguish between, *e.g.*, the size of a cone that is one meter tall, and a cone that is 100 meters tall, each with the same base. In fnite-dimensional Euclidean space, it is most natural to take the size of a region to be its standard Lebesgue *measure*.

Let us then restrict our attention to a simple case: the real line (or one-dimensional Euclidean space). How to construct a measure function on the algebra of regular closed subsets of this space? As we noted already, each element in the algebra contains one regular closed representative set. Since this set is Borel, it is measurable. Thus we can assign to each element of the algebra the Lebesgue measure of its unique regular closed representative. But here is where we run into diffculties.

[37]See (35), p. 6.

The measure of a countable fusion (or, in algebraic terms, join) of disjoint regions of space should equal the sum of their individual measures. This is the principle of Countable Additivity. But the measure function just defned violates this constraint. To see this, consider a thick, or 'fat', Cantor set. We construct the set in stages, starting with the real interval, [0, 1], and at the frst stage of construction, removing the open middle $\frac{1}{4}$ of that interval. We are now left with the intervals $[0, \frac{3}{8}]$ and $[\frac{5}{8}, 1]$. At the second stage of construction, we remove the open middle $\frac{1}{1}$ of these remaining intervals. In general, at stage n we remove the open middle $(\frac{1}{4})^{n+1}$ of all remaining intervals from the previous stage ($n \geq 0$).[38] [39] The sum of the measures of the removed intervals is

$$\sum_{n\geq 0} 2^n \left(\frac{1}{4}\right)^{n+1} = \frac{1}{4} \sum_{n\geq 0} \left(\frac{1}{2}\right)^n = 1/2$$

but the union of these intervals is equivalent, in the algebra of regular closed sets, to the entire interval, which has measure equal to 1.

How serious a problem is this? Unfortunately, moving to a different measure on the algebra will not help matters. It can be shown that *any* measure defned on every element of the algebra of regular closed subsets of reals is not countably additive. We could, perhaps, look to measures that are only defned on *some* elements of the algebra, but even this does not look promising. After all, we must at the very least have measures for intervals, and there really is no natural alternative to identifying the measure of an interval (or an element of the algebra represented by an interval) with its length. But the example just given shows that already at the level of intervals, countable additivity fails. It seems right to conclude with F. Arntzenius that "our attempt to do physics in this kind of pointless topological space is in big trouble. (1, p. 18)

1.6.3 The measure-theoretic approach

In recent work, Arntzenius proposes an alternative, measure-theoretic approach to modeling gunky space, which makes signifcant revisions to Whitehead's program.[40] Arntzenius, like Whitehead, takes as primitive a relation of 'connectedness' among regions of space, but his aim is to allow for models in which we have

[38] The set constructed here is the complement of the *Smith-Volterra-Cantor set*, and has measure $\frac{1}{2}$. This is not mandatory. An easy manipulation of the lengths of intervals in the construction yields a Cantor set of measure arbitrarily close to zero or one. For a fuller discussion of the Smith-Volterra Cantor set, see (42).

[39] The intervals removed are open intervals, as is standard in the Cantor construction. But of course, each such interval is identifed in the algebra with its closure, which is a regular closed set and has the same measure.

[40] See (1).

a workable notion of measure as well. On Arntzenius's approach, instead of identifying sets in pointy space if the closures of their interiors are equal (as we do in the algebra of regular closed sets), we identify sets that differ from one another by a set of Lebesgue-measure zero. Thus gunky space is modeled not via algebras of regular closed sets, but via measure algebras arising from pointy topological spaces together with Borel measures. To simplify matters, let's focus for the moment on one dimensional Euclidean space with standard Lebesgue measure. The algebra corresponding to this space is just the *Lebesgue measure algebra* defned above — the very same algebra used to give a probabilistic semantics for modal languages. The frst thing to note is that in this algebra individual points disappear. Indeed, individual points have measure zero, so *modulo measure zero* any singleton set is identifed with the empty set, and so represents the bottom element of the algebra.[41] Moreover, like the algebra of regular closed sets, the Lebesgue measure algebra is *atomless*: every non-zero element of the algebra dominates some other, non- zero element. Again, this means that our model satisfes the thesis GUNK (S) given above. These two facts are no doubt congenial to a gunky point of view. But the real advantages in turning to the measure-theoretic approach are that here, unlike in Whiteheadian space, we can defne a workable notion of measure. Indeed, since representative sets in a given equivalence class in the algebra differ from one another by a set of measure zero, we can defne the measure of an element in the algebra to be the measure of any of its representative sets. This measure function is countably additive. Moreover, only the bottom element of the algebra — that element represented by the empty set — has measure zero. In words, every region of space has a precise size, and no region, except the null region, has size equal to zero. So far, things seem quite promising.

But in addition to mereological and measure structure, we would like space to have *topological* structure, and here is where certain complications arise.[42] Standard topological structure is, as we know, defned in terms of a collection of primitively distinguished open (or closed) sets (a *closed* set is the complement of an open set). In Euclidean space, for example, basic open sets are open spheres, or spheres without any of the points on their surface. But according to Arntzenius, Hawthorne, and Russell,[43], the distinction between open and closed sets is one

[41] More generally, for any point, a, and subset, A, of Euclidean space, the sets A � $\{a\}$ and $A \cup \{a$ are equivalent in the Lebesgue measure algebra.

[42] I take mereological structure to be Boolean structure, or that structure captured by the ordering relation, '$\preceq$' on the algebra. A more careful presentation of this material would state explicitly which mereological assumptions are made, but in this short introduction we do not have the space to spell out the details. For a fuller discussion, see (35).

[43] See (1), (2), and (35).

which cannot be made in the setting of the Lebesgue measure algebra:[44]

> "Mathematical orthodoxy casts topological structure in terms of primitively distinguished open point-sets. But among the spaces we are concerned with here are those that make no distinction between closed and open regions; so the orthodox approach won't do." (35, p. 253)

> "The topological structure we will give pointless regions can not be given in the same way that we gave pointy spaces topological structure, namely in terms of a distinction between open and closed regions. For that is exactly the kind of distinction that we do not believe exists if reality is pointless. " (1, p. 237)

Why is it that, according to these philosophers, there can be no distinction between open and closed regions on the measure-theoretic approach to gunk? Many open sets differ from their closure by a set of measure zero (where the *closure* of a set is the smallest closed set containing it). Consider, for example, any open interval. This set differs from its closure only at the endpoints. If sets of measure zero do not exist, then the distinction between such an open set and its closure would seem to collapse: these two regions of space could not be told apart. The conclusion these authors have drawn is that if we are to have topological structure on the Lebesgue measure algebra, it must be topological structure of a non-standard variety—topology done, not in terms of primitively distinguished open (or closed) sets, but in terms of other primitive notions that do not rely on the existence of points or sets of measure zero.

The adoption of non-standard topological primitives is not itself anything new. Indeed, Whitehead did this in 1929, when he took as primitive the binary relation of 'connectedness,' and used this to axiomatize all of pointless geometry. Some years later, A. Grzegorczyk assumed as primitive the relation of being separated, providing an axiomatization of that relation which allowed him, like Whitehead, to defne points.[45] More recently, and in the same tradition, P. Roeper takes as primitive both the relation of connectedness and the property of being limited, and axiomatizes these notions by way of defning what he calls 'region-based topology.'[46] In taking the relation of connectedness and the property of being limited as primitive, Arntzenius attempts to show that the measure-based approach can, in

[44]One should be careful not to read too much into talk of the *distinction* between open and closed sets. Depending on the topological space, we may have sets that are both open and closed. Finite-dimensional Euclidean space (R^n) is connected, and so in this special case there are no sets of this kind. In general, however, this is not the case.

[45]See (14).

[46]See (34).

some sense, mimic the topological structure that Roeper gives for Whiteheadian space: that Roeper's axioms are satisfed not just in regular closed algebras, but also, to a large degree, in measure algebras.[47]

But is the turn to non-standard topological primitives necessary in the measure-theoretic setting? Let us re-examine the arguments given against standard topological structure in more detail. As Arntzenius and Hawthorne argue:

> "...When No Zero is combined with our mereological assumptions, further results follow. In standard point-set topology, we can distinguish an open region from its closure. Typically, each has the same volume, since the latter differs from the former only by including the boundary points of the former. Can the Gunk lover admit a distinction between such closed and open parts...?
>
> Assume for reductio there is some open piece, call it 'Open', that is a proper part of some closed piece, call it 'Closed', each of the same volume. Remainder[48] tells us that there will be a part x of Closed that does not overlap with Open, such that Closed is the fusion of x and Open. Assuming Finite Additivity, it follows that x has zero measure, violating No Zero. *So, once No Zero is assumed, we cannot admit the standard distinction between open and closed regions*." (2, p. 443)[49]

Summarizing the argument: Because *in many cases* an open set differs from its closure by a set of measure zero, there can be no distinction between open and closed regions.

I now want to argue that such arguments fall fat. While turning to non-standard topological primitives makes sense in the context of Whiteheadian space (where in some sense every element is open, and there are no boundary regions), this is not the case for the measure-theoretic setting, where space is not topologically distinctive in the same ways. Of course, in the Lebesgue measure algebra, there is no distinction between sets that differ by a set of measure zero. So if an open subset of the reals and its closure have the same measure, then these two sets are identifed. This is the case for many familiar subsets of the real line: for example, any interval, or fnite union of intervals. But it does not follow that we have to throw out the distinction between open and closed regions altogether. In the measure-based semantics for modal languages, we defned an open element of the Lebesgue measure

[47]For Roeper's axioms, see Appendix A.

[48]The Remainder principle states: If x is a part of y and not identical to y then there is some z that is part of y that is discrete from x, such that y is the fusion of x and z (where x is discrete from y iff there is no part that x shares with y).

[49]My emphasis.

algebra to be any element that has an open representative, or representative that is an open subset of the real interval, [0,1]. Likewise, let us defne a *closed* element of the algebra to be the Boolean complement of an open element.[50] An immediate question presents itself: Are there any elements of the algebra that are *not* open? (Equally: are there any elements of the algebra that are *not* closed?) Consider the thick Cantor set mentioned above. The element of the Lebesgue measure algebra, c, corresponding to this set is not open. Indeed, as we show in Chapter 2, the thick Cantor set differs from every open subset of reals by a set of non-zero measure. Moreover, the same example shows that there are open elements of the algebra that are *not* equal to their own closure. Indeed, the complement of c is an open element of the algebra that has measure strictly less than 1,[51] and its closure is the top element in the algebra. Here, then, we have a non-trivial algebraic distinction between open and closed regions of space: precisely the sort of distinction with which to do standard topology.

One may object at this point that the thick Cantor set and its complement are rather *special* sets. "According to the defnitions given," you say, "*most* elements of the algebra are both open and closed and so the distinction between 'open' and 'closed' in the algebra cuts little water." There are two ways to respond. First, although it is true that many of the open subsets of the real line that we talk about in mathematics differ from their closure by a set of measure zero (hence are identifed with their closure in the Lebesgue measure algebra), the sets that we tend to *talk* about are a very restricted few. By necessity, such sets are ones that can be simply described. But limits on our discursive powers should not mislead us as to the variety of subsets of the real line. There are many sets that we are not accustomed to talk about because they are not easy to defne, but which exist all the same. (In a certain sense, even the thick Cantor set is quite simple. It has a very regular, fractal structure.) So while it may be true that for many *familiar* subsets of the real line, the distinction between closed and open collapses once we move to the Lebesgue measure algebra, there is, I think, no sense to the notion that 'most' subsets are like this. But second, if space really is pointless, then we should expect to modify our view of space in sometimes signifcant ways. The distinction between simple open regions of the real line (*e.g.*, intervals) and their closures must, of course, fall by the wayside. Another way to put the fact that such regions do not differ from their closure, is to say that they have no *boundary*. Now the boundary of an open interval in the real line is just the endpoints of the interval. Surely on a gunky conception of space—a conception of space on which there are no point-

[50]Equivalently, a *closed* element is any element that has a closed representative, or representative that is a closed subset of the real line.

[51]Hence, the complement of c is not equal to the top element in the algebra.

sized parts—we should deny that such regions exists. But unlike in Whiteheadian space, there are regions of space that *are* properly called 'boundary.' Indeed, the thick Cantor set is one example. It has non-zero measure, and yet has no interior. On the measure-theoretic approach to gunk, this result is quite welcome. It is not that boundary regions do not exist, but rather that *true* boundary regions—regions that do not simply consist of the endpoints of intervals—are quite special.

Still, many questions remain. Arntzenius and Roeper take as primitive the relation of connectedness and the property of being limited. These notions have some intuitive appeal. It would be nice if we could reproduce them in the measure-theoretic setting without taking them as primitives—either by defning those relations in terms of the open-closed distinction, or by adding additional topological structure to our measure models. In Appendix A we suggest a way to this this according to the second approach. (By the isomorphism results of Chapter 4, the frst approach will not work.) Further questions concern our ability to extend these defnitions to reduced measure algebras that do not arise from Euclidean spaces. Unfortunately, we do not have the time or the space to pursue those questions here. I hope, at any rate, that these loose remarks point in a direction in which this work will be further developed.

1.7 Game plan

The dissertation is organized as follows. In Chapter 2, we develop in detail the topological semantics, and show that Tarski's completeness result for the real line can be proved in a simplifed way, using well-known fractal curves. In Chapter 3, we develop the probabilistic semantics, and prove that $S4$ is complete for this semantics. Also in this chapter, we show that intuitionistic propositional logic (IPC) is complete for the subframe of open elements in the Lebesgue measure algebra. In Chapter 4, we show that the probabilistic semantics can be extended to dynamic topological logics—or multimodal logics intended to describe dynamic spaces. Here we prove that $S4C$ is complete for the probabilistic semantics and develop some interesting isomorphism results that allow us to extend completeness to other measure algebras. The reader interested in some, but not all, of the results that follow is invited to skip ahead to the relevant chapter. Individual chapters are written so as to be readable independently of one another.

Chapter 2

Topological Semantics for Modal Logic: The Tarski Theorem Reproved

Abstract. This chapter explores the connection between fractal geometry and topological modal logic. In the early 1940's, Tarski showed that the modal logic $S4$ can be interpreted in topological spaces. Renewed interest in Tarski's topological semantics can be seen in such recent papers as (5), (18), (39), and (40). In this chapter we introduce the use of fractal techniques for proving completeness of $S4$ and non-trivial extensions of $S4$ for a variety of spaces in the topological semantics. These techniques are developed to relate the somewhat peculiar non-Hausdorff tree topologies with more familiar Euclidean and other metric topologies. The main results of the chapter are completeness of $S4$ for the binary tree with limits, and completeness of $S4$ for the Koch Curve, a well-known fractal curve. An important corollary is a new and simplifed proof of completeness of $S4$ for the real line, R (originally proved by Tarski and McKinsey in (27)).

2.1 Introduction

[1] In the late 1930's, Tarski developed a topological semantics for modal logic in which formulas are interpreted in topological spaces. In a topological model, each propositional variable, P, is assigned to an arbitrary subset of a given topological space—the set of points where P is true. Conjunctions, disjunctions and negations are interpreted as set-theoretic intersections, unions and complements; the 'necessity,' or 'D'-modality is interpreted as a topological interior operator. (Thus, 'Dφ' is true throughout the interior of the set of points where 'φ' is true.) Although this semantics was largely forgotten in the years since Kripke's relational semantics was introduced, the last ffteen years have witnessed a burst of renewed interest. Indeed, researchers have come to see Tarski's work as the foundation of the much broader project of using modal logic to describe space and spatial structures. As early as 1944, Tarski and McKinsey showed that the modal logic $S4$ is sound and complete for any dense-in-itself metric space (27). Their proof was notoriously complex, and in recent years, completeness for the special case of the real line was reproved in such papers as (5), (18), (26), (29), and (38). In this chapter, we explore new, fractal techniques for proving a variety of completeness results in the topological semantics.

The main result of the chapter is a proof of completeness of $S4$ for the *Koch Curve*, a well known fractal curve. An important corollary is a new proof of completeness of $S4$ for the real line, R. The fractal techniques introduced in these proofs are, as we argue, the chapter's main contribution to the topological semantics for modal logic. The results of Section 4 and the techniques developed below are not tailor-made for solving completeness of $S4$ for the real line or for the slightly wider problem of completeness of $S4$ with respect to interesting classes of metric topological models. The main technique is developed to relate formally the somewhat peculiar non-Hausdorff tree topologies with more familiar Euclidean and other metric topologies. As we will see, completeness is transferred from an appropriate tree to a metric space by means of a known fractal curve. Completeness for both the Koch Curve and R are best seen as examples of the power of the fractal techniques introduced.

The chapter is organized in fve sections. Section 1 introduces the basic propositional modal language and Kripke (relational) semantics, and recalls some basic completeness results. Section 2 demonstrates the use of trees as Kripke frames, and shows that $S4$ is complete for the infnite binary tree. Section 3 explores the topological semantics for the modal language, introduces the complete binary tree

[1]This chapter is a slightly modifed version of a paper co-authored with Darko Sarenac, "Fractal Completeness Techniques in Topological Modal Logic." See (23).

(or infnite binary tree with limits), and shows that $S4$ is complete for this tree. Section 4 is the part of the chapter where we prove our main results. In this section we introduce the Koch Curve, and simultaneously prove completeness of $S4$ for the Koch Curve and for the real interval $[0, 1]$. The reader familiar with modal logic can skim through much of Sections 1 and 2. Furthermore, the reader familiar with Tarski's topological semantics can leaf through all but the proof of completeness of $S4$ for the complete binary tree in Section 3. If the reading seems somewhat terse in places, suffcient background information can be obtained by reading the excellent and very current summary of the state of topological modal logic in (40).

2.2 Kripke semantics for $S4$

2.2.1 Language, models, and truth

Let the modal language L consist of a countable set, $\mathsf{P} = \{P_i \mid \text{ for all } i \in \mathsf{N}\}$, of atomic variables and be closed under binary connectives $\rightarrow$, $\vee$, $\wedge$ and unary operators $\neg$, D, *3*.

A *frame* is an ordered pair, $\mathsf{F} = \mathsf{h}U, R\mathsf{i}$, where U is a set of points called the *universe*, and R is a binary relation on U. We say F is *transitive* (refexive) if R is transitive (refexive). We interpret L in a *model* $M = \mathsf{hF}, V\mathsf{i}$, where F is a frame, and $V : \mathsf{P} \rightarrow \mathsf{p}(U)$ is a *valuation* function.

Formulas are interpreted on points $x \in U$ and we write $M, x = \varphi$ to mean that in the model M at the point x, φ holds. More specifcally for a model $M = \mathsf{hh}U, R\mathsf{i}, V\mathsf{i}$ and a point $x \in U$, the ternary relation $M, x \models \varphi$ is interpreted inductively as follows. For $P \in \mathsf{P}$,

$M, x \models P \Leftrightarrow x \in V(P)$
$M, x \models (\varphi \vee \psi) \Leftrightarrow M, x \models \varphi$ or $M, x \models \psi$
$M, x \models \neg\varphi \Leftrightarrow M, x \models \varphi$
$M, x \models \mathrm{D}\varphi \Leftrightarrow M, y \models \varphi$ for all y such that Rxy
$M, x \models 3\varphi \Leftrightarrow M, y \models \varphi$ for some y such that Rxy.

The interpretation for $\wedge$, $\rightarrow$ and $\leftrightarrow$ can be obtained from the above via the standard defnitions. We could have defned $3P$ as $\neg\mathrm{D}\neg P$ but the defnition was added for the completeness of presentation.

Defnition 2.2.1 (Logic $S4$). *The modal logic $S4$ in the language L consists of some complete axiomatization of classical propositional logic PL, some complete axiomatization of the minimal normal modal logic K, say the axiom:*

$C : (DP \wedge DQ) \to D(P \wedge Q)$,

and the rules:

RN: $\vdash \varphi \Rightarrow \vdash D\varphi$, *and*

RM: $\vdash \varphi \to \psi \Rightarrow \vdash D\varphi \to D\psi$;[2]

and, fnally, the special S4 axioms:

$4 : DP \to DDP$

$T : DP \to P$

We defne standard validity relations. Let $\mathsf{F} = (U, R)$ be a frame, and let $M = (\mathsf{F}, V)$ be a model over F. For any formula $\varphi \in L$, we say φ is *true* in M if $M, x \models \varphi$ for all $x \in U$. We say φ is *valid in* F if φ is true in every model over F. If $\mathcal{C}$ is a class of frames, we say φ is *valid in* $\mathcal{C}$ if φ is valid in every frame in $\mathcal{C}$. Finally, the logic $S4$ is *complete* for $\mathcal{C}$ if every formula valid in $\mathcal{C}$ is a theorem of $S4$ (*i.e.*, can be derived from the axioms together with the rules of inference). With slight abuse of notation, we will sometimes say that $S4$ is complete for for a single frame F, where we mean $S4$ is complete for $\{\mathsf{F}\}$.

2.2.2 Kripke's classic completeness results

Defnition 2.2.2 (Rooted Frames and Models)**.** *A* rooted *(or* pointed*) frame is a triple,* $\mathsf{F} = (U, R, x)$*, where* (U, R) *is a frame,* $x \in U$*, and for all* $y \in U$*,* $(x, y) \in R$.

That is, the point x is R-related to every other point in U (or x "sees" all $y \in U$, for short).

Theorem 2.2.3. *[Kripke]*

The modal logic S4 is sound and complete for (i) the class of all transitive, refexive frames; (ii) the class of all fnite transitive, refexive frames; (iii) the class of all rooted, fnite, transitive, refexive frames.

We will not reproduce this classic result here. Most standard introductory presentations of modal logic contain proofs of (*i*), (*ii*), and (*iii*). For Kripke's original proof we refer the reader to (20); for a more contemporary variant, see (7).

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[2] This somewhat unusual axiomatization of K and hence of $S4$ makes the topological connection introduced later on in the chapter more explicit. C interpreted topologically states that the intersection of opens is open, RN states that the universe is open, RM states that if P is a subset of Q, then the interior of P is a subset of the interior of Q. Furthermore, T states that the interior of P is a subset of P, and, fnally, 4, together with T, states that the interior of the interior of P is just the interior of P. This should strongly remind the reader of Kuratowski's axiomatization of the interior operator.

In the next section we recall that the infnite binary tree, T_2, with a transitive, refexive relation, R_2, can be used to build models for the modal language. Indeed, the logic $S4$ is complete for the class of models over the frame T_2: a modal formula φ is a theorem of $S4$ if and only if it is valid in every model over T_2. Below, we show how to view T_2 (and, for that matter, *any* transitive, refexive frame) as a topological space. We then introduce an uncountable topological extension of T_2 that we call T_2^+. This new structure extends T_2 by adding to it uncountably many "limit nodes," corresponding to each (infnite) branch of T_2. Our main contribution to the theory of tree topologies is the proof that $S4$ is complete for T_2^+. As we mentioned above, the signifcance of T_2^+ for us lies in large part in its use in extending topological completeness results to various metric and fractal spaces. We start with a brief discussion of T_2 viewed as a relational frame.[3]

2.3 Infnite binary tree

2.3.1 The modal view of the infnite binary tree, T_2

Let $\Sigma = \{0, 1\}$, and let Σ^* be the set of all fnite strings over Σ including , the empty string. Let Σ^ω be the set of all countably infnite strings over Σ, and let $\Sigma^+ = \Sigma^* \cup \Sigma^\omega$. For $x, y \in \Sigma^*$, let $x * y$ denote the concatenation of x and y. We will also write xy for $x * y$. Concatenation is further defned for $x \in \Sigma^*$ and $y \in \Sigma^\omega$, but not for $x, y \in \Sigma^\omega$.

Note that Σ^* is closed under concatenation, that is, if $x, y \in \Sigma^*$ then $x * y \in \Sigma^*$. Similarly, Σ^+ is closed under "right-concatenation" in the following sense: for $x \in \Sigma^*$, $y \in \Sigma^+$, $x * y \in \Sigma^+$.

We let $s_i : \Sigma^* \to \Sigma^*$ for $i \in \{0, 1\}$ be the function defned by $s_i(x) = x * i$. Thus for example $s_0(1) = 10$, and $s_1(110) = 1101$. We call $s_0(x)$ the "left successor" of x and $s_1(x)$ the "right successor" of x.

We can now defne the binary relation R_2 on Σ^* as the transitive refexive closure of $s_0 \cup s_1$ (where s_i is viewed here as a relation, rather than a function).

Defnition 2.3.1 (T_2, a modal frame). $T_2 = \langle \Sigma^*, R_2, \langle\cdot\rangle\rangle$

We call T_2 the *infnite binary branching tree* or *full binary tree*. We call the empty string, $\langle\cdot\rangle$, the *root*, and for any $x \in T_2$, $s_0(x)$ and $s_1(x)$ are called the *immediate successors* of x. For simplicity of notation, we will often leave out the root, $\langle\cdot\rangle$, denoting T_2 by $\langle \Sigma^*, R_2 \rangle$.

[3] The formal details of the next section follow the presentation in (40). The details can be skipped by a reader familiar with the notion of *tree unravelling*.

Fact 2.3.2. *Every node x is accessible from the root in finitely many steps along R_2 and hence in one step by transitivity. Every $x \in T_2$ has exactly two immediate successors and countably many successors altogether.*

A valuation function $V : P \to \wp(\Sigma^*)$ defines a model $\mathcal{T}_2$ over T_2. Since T_2 is transitive and reflexive any such model validates the $S4$ axioms—*i.e.*, $S4$ is sound for T_2.

Claim 2.3.3. *For any finite, transitive, reflexive, rooted, model M, with root x, there is a valuation V over T_2 such that,*

$$M, x \models \varphi \Leftrightarrow \langle T_2, V \rangle, \langle \cdot \rangle \models \varphi$$

for every $\varphi \in L$.

(The proof of the claim is postponed until the next section.)

It follows from Claim 2.3.3 and Theorem 2.2.3 that every nontheorem of $S4$ can be shown false on some model based on the frame T_2. Indeed, if φ is not a theorem of $S4$, then by Theorem 2.2.3, there is some finite rooted frame $\mathsf{F} = \langle U, R, x \rangle$ and valuation V such that $\langle \mathsf{F}, V \rangle, x \not\models \varphi$. But then by Claim 2.3.3, there is a valuation V' over T_2 such that $\langle T_2, V' \rangle, \langle \rangle \not\models \varphi$. Thus any nontheorem fails on T_2, and $S4$ is complete for the class of models over T_2.

2.3.2 Building a p-morphism from T_2 onto finite Kripke frames

We prove Claim 2.3.3 by constructing a p-morphism $f : T_2 \to \mathsf{F}$, where $\mathsf{F} = (U, R, x)$ is a finite, rooted, transitive and reflexive frame. We briefly recall the notion of p-morphism.

Definition 2.3.4 (p-morphism)**.** *Let $\mathsf{F} = \langle U, R, x_r \rangle$ and $\mathsf{F}' = \langle U', R', x'_r \rangle$ be rooted frames. A p-morphism from F to F' is a function $f : U \to U'$ satisfying: For any $x, y \in U$ and $y' \in U'$,*

(i) $f(x_r) = x'_r$;

(ii) If Rxy, then $f(x)R'f(y)$;

(iii) If $R'f(x)y'$, then there is a $z \in U$, Rxz and $f(z) = y'$.

We say that f is a surjective p-morphism *if, in addition, $f(U) = U'$.*

Fact 2.3.5. *If there is a surjective p-morphism f from F to F', then for any valuation function $V : \mathsf{P} \to \wp(U')$, any point $x \in U$, and any modal formula φ, we have:*[4]

[4]The function $[f^{-1}] : \mathsf{P}(U') \to \mathsf{P}(U)$ raises the type: for $A \subseteq U'$, $[f^{-1}](A) = \{y \mid f(y) \in A\}$. Note that although f^{-1} is likely not a function, $[f^{-1}]$ is always a function, but of a higher type. Thus, the function $[f^{-1}] \circ V : \mathsf{P} \to \mathsf{P}(U)$, i.e., it is a valuation function.

$$\langle F, [f^{-1}] \circ V \rangle, x \models \varphi \Leftrightarrow \langle F', V \rangle, f(x) \models \varphi$$

Thus, to prove Claim 2.3.3 it suffices to show that for any finite, transitive, reflexive, rooted frame $F = \langle U, R, x \rangle$, there is a surjective p-morphism f from T_2 to F.

Let the cardinality of U in F be n. Notice that no point in U has more than n distinct successors and x, the root, actually has n successors. We now construct the function f. For $1 \leq i \leq n (= |U|)$, we define the set of functions $s_i : U \to U$ $(1 \leq i \leq n)$. For each $y \in U$, the function s_i chooses the ith distinct R-successor of y, if such a successor exists. Otherwise $s_i(y) = y$. More formally,

Definition 2.3.6 (Successor functions s_i). *For all y, $s_1(y) = y$ (s_1 is the identity function). Fix $i \in \mathbb{N}$, and suppose that $s_1(y)$, $s_2(y)$, …, $s_{i-1}(y)$ are already defined, and that $Rys_k(y)$ for all $k < i$. Then we let $s_i(y)$ be some $z \in U$ such that Ryz and $s_k(y) \neq z$ for all $k < i$, if there is some such z. Else, $s_i(y) = y$.*

Example 2.3.7 (A set of successor functions). *Let $y \in U$ have 3 distinct successors including y itself: y, w and z and no others. Then if $|U| = 5$, we let $s_1(y) = y$, $s_2(y) = w$, $s_3(y) = z$, but $s_4(y) = s_5(y) = y$ as we have run out of distinct successors.*

Definition 2.3.8. *[UNRAVELING p-MORPHISM]*
We define a linear ordering on the nodes in T_2. This can be done in many ways, but for specificity, we let, e.g., $\langle\cdot\rangle < 0 < 1 < 00 < 01 < 10 < 11 < 000 < \ldots$
[BASE STEP.] First let $f(\langle\cdot\rangle) = x$.
[RECURSIVE STEP.] Until f is defined for all nodes in T_2, find the least[5] node t such that $f(t)$ is defined, but neither $f(t * 0)$ nor $f(t * 1)$ is defined. Assume that $f(t) = y$. Then let,

$$f(t*1) = s_1(y),\ f(t*01) = s_2(y),\ f(t*001) = s_3(y),\ \ldots\ f(t*0^{n-1}*1) = s_n(y)$$

where 0^{n-1} is a sequence of $n - 1$ zeros. Finally, let,

$$f(t * 0) = f(t * 00) = f(t * 000) = \ldots = f(t * 0^n) = s_1(y) = y.$$

Lemma 2.3.9. *[Unravelling Lemma] Let f be the function defined in Definition 2.3.8. Then f is a p-morphism.*

[5] On the ordering just given.

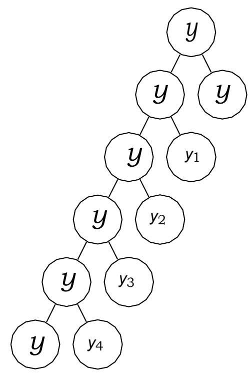

Figure 5: The recursive step of the defnition of the p-morphism f. Here $U = 5$, $f(t) = y$, $s_1(y) = y$, $s_2(y) = y_1$, $s_3(y) = y_2$, $s_4(y) = y_3$, and $s_5(y) = y_4$. Following the defnition, $f(t * 01) = y_1$, $f(t * 001) = y_2$, $f(t * 0001) = y_3$, $f(t * 00001) = y_4$, and all other points visible in the diagram are labeled y. No successor of t except for the eleven nodes (really ten and t) explicitly shown in the diagram is labeled at this stage.

Proof. (*i*) It suffces to show that if $R_2 st$ and t is the immediate successor of s, then $Rf(s)f(t)$. This can be seen by inspecting the recursive step of Defnition 2.3.8. If $f(s) = y$, then $f(t)$ is $s_i(y)$, for some $i \in \{1, \ldots, n\}$, but, by defnition of s_i, we know $Rys_i(y)$ for each such i. (*ii*) We need to show that if $Rf(t)z$, then there exists $s \in T_2$ such that $R_2 ts$, and $f(s) = z$. We let $f(t) = y$ and recall that $s_1(y), s_2(y), \ldots, s_n(y)$ exhaust the distinct $\underline{R}$ successors of y in $\mathcal{F}$. Then for some $i \in \{1, \ldots, n\}$, $s_i(y) = z$. If t was ever the least node satisfying the antecedent condition of Defnition 2.3.8, then some successor of t was labeled by $s_i(y)$—*i.e.*, by z. Otherwise, t is a successor of some other node t', which did at some stage satisfy the antecedent condition of Defnition 2.3.8 and $t = t' * 0^k$ for some $k \leq n$. But then, at that stage, for some successor t'' of t, $f(t'') = y$ and $t'' * u$ was undefned for any nonempty fnite sequence u. Thus at some future stage a successor of t'' was labeled with $s_i(y)$ (i.e. z). But a successor of t'' is a successor of t by transitivity of R_2, as desired. □

Putting Fact 2.3.5 and Lemma 2.3.9 together, we obtain the desired completeness result:

Fact 2.3.10. *The modal logic $S4$ is complete for the class of models over the frame* $T_2 = (\Sigma^*, R_2, \langle \cdot \rangle)$.

In the next section we look at modal language L and the frame $T_2 = (\Sigma^*, R_2)$ from a topological perspective.

2.4 Topological semantics for $S4$

We now turn to topology and the topological interpretation of the modal language L. Long before Kripke-semantics for the modal language was established as the yardstick, A. Tarski and J.C.C. McKinsey noted an irresistible connection between Lewis and Langford's axioms for the modal logic $S4$, and Kuratowski's axioms for the topological interior operator. The topological interpretation of modal logic exploits this connection.[6]

Tarski's idea was to view $\Box A$ as the interior of the set A and $\Diamond A$ as the closure of A and try to understand what kind of logical structure such an interpretation supported. Tarski was able to prove—in some sense quite unsurprisingly—that under this interpretation the logic of the interior and closure operators turns out to be nothing less than $S4$. The argument for the general case is straightforward, as we'll see below. The arguments for specifc topological spaces turn out to be rather more involved. It is part of our goal here to try to understand where such complexity comes from. Let us introduce some basic background notions.

2.4.1 Topological semantics

A topology is a set of points with some spatial structure (one can think of it as a set of points glued together in a certain way). Specifcally, a topology is a pair, $\langle X, \tau \rangle$, where X is a set and $\tau \subseteq \mathcal{P}(X)$ satisfes,

1. $X, \emptyset \in \tau$,
2. If $A, B \in \tau$, then $A \cap B \in \tau$,
3. If $A_i \in \tau$ for all $i \in I$, then $\bigcup_{i \in I} A_i \in \tau$

If in addition a topology satisfes,

4. If $A_i \in \tau$ for all $i \in I$, then $\bigcap_{i \in I} A_i \in \tau$

then the topology is called *Alexandroff*. As we'll see, most interesting topologies are *not* Alexandroff. More (structure) is not always better, as a cursory comparison between Italian and American pizza quickly reveals.

Although a topological space is strictly speaking a pair, (X, τ), we will for simplicity of notation (and where the meaning is clear) often denote both the topological space itself and the underlying set of points by X. The sets in τ are called

[6] Equivalently, one can exploit the connection between the $\Diamond$-version of the $S4$ axioms and the behavior of the *closure* operator C, via the defnition $I(A) = -C(-A)$. (In words, the interior of a set is the complement of the closure of the complement of that set.)

open sets. We say a set is *closed* if its complement is open. The union of open subsets of a set, A, is called the *interior* of A:

$$Int(A) = \bigcup \{O \text{ open} \mid O \subseteq A\}$$

The *closure* of a set is the complement of the interior of the complement:

$$Cl(A) = -Int-(A)$$

(Equivalently, a point x is in the closure of A if every open set containing x contains some element in A.)

We wish to interpret our language L in topological models. A topological model is a pair $M = \langle X, V \rangle$ where X is a topology and $V : P \to \mathcal{P}(X)$ is a valuation function. We define a ternary relation $M, x \models \varphi$ that as before holds between a point in a model and a formula. The cases for the atomic and Boolean formulas are the same. The only real difference is in the modal cases of $\Box$ and $\Diamond$. We want $\Box\varphi$ to be true at a given point x if x is in the interior of the set defined by the formula φ. Then also $\Diamond\varphi$ should hold at x if x is in the closure of the set defined by φ. We encode these observations in the following truth definitions:

$M, x \models \Box\varphi \Leftrightarrow \exists O$ open such that $x \in O$ and $\forall y \in O, M, y \models \varphi$.
$M, x \models \Diamond\varphi \Leftrightarrow \forall O$ open, $x \in O$ implies $\exists y \in O$ such that $M, y \models \varphi$.

Let X be an Alexandroff topology and let $x \in X$. Consider the set $O_x = \bigcap\{O \text{ open} \mid x \in O\}$, *i.e.*, the intersection of all open sets containing x. Note that since our topological space is Alexandroff, this is a non-empty open set. We define the binary relation R on X:

$$Rxy \Leftrightarrow y \in O_x.$$

Claim 2.4.1. $\mathcal{F}_X = \langle X, R \rangle$ *is a reflexive, transitive frame.*

Proof. For reflexivity, note that $x \in O_x$. For transitivity, suppose Rxy and Ryz. Then $y \in O_x$ and $z \in O_y$. From the first inclusion it follows that $O_y \subseteq O_x$. So we have $z \in O_y \subseteq O_x$, and hence Rxz. □

Moving in the reverse direction, we can generate a topology from a reflexive, transitive frame. Let $\mathcal{F} = \langle X, R \rangle$ be a reflexive, transitive frame. We will say that a subset O of X is *open* if it is upward-closed under R (where a set O is upward-closed under R if $x \in O$ and Rxy implies $y \in O$). Note that the collection of open sets are closed under finite intersections, arbitrary unions, and contain both the empty set and the entire space X. Let $X_{\mathcal{F}}$ be the topological space defined in this way. Then,

Claim 2.4.2. *X_F is Alexandroff.*

Proof. The reader can verify that X_F is a topological space. To see that it is Alexandroff, suppose that $\{O_i \mid i \in I\}$ is a collection of open sets in the topology and let $x \in \bigcap_{i \in I} O_i$ and Rxy. Then since each O_i is upward-closed under R, $y \in O_i$ for each $i \in I$. But then $y \in \bigcap_{i \in I} O_i$, and $\bigcap_{i \in I} O_i$ is upward-closed under R, as desired. □

The reader is invited to verify that the operations of generating a transitive, refexive frame from an Alexandroff topology, and of generating an Alexandroff topology from a transitive, refexive frame just described are inverses of one another: if one starts with an Alexandroff topology, then generates a transitive refexive frame, and then, from this frame, generates an Alexandroff topology in the manner described, one ends up with the original topological space (and similarly, when one starts from a transitive, refexive frame). When a frame and topological space are generated in this way by one another, we will sometimes say they "correspond." The next proposition states that corresponding frames and topological spaces satisfy the same modal formulas:

Proposition 2.4.3. *Let X be an Alexandroff topology and let F be a transitive, refexive frame. If X and F correspond, then for any formula φ in L, any $x \in X$, and any valuation $V : P \to P(X)$,*

$$\langle F, V \rangle, x \models \varphi \iff \langle X, V \rangle, x \models \varphi$$

Proof. The proof is by induction on the complexity of φ. We show only the modal clause, $\varphi :\equiv \Box\psi$. We have,

$$\begin{aligned}
\langle F, V \rangle, x \models \Box\psi &\iff \langle F, V \rangle, y \models \psi \text{ for all } y \text{ such that } Rxy \\
&\iff \langle F, V \rangle, y \models \psi \text{ for all } y \in O_x \\
&\iff \langle X, V \rangle, y \models \psi \text{ for all } y \in O_x \qquad \text{(by IH)} \\
&\iff \langle X, V \rangle, x \models \Box\psi
\end{aligned}$$

□

What these observations tell us is that Alexandroff topologies are nothing more than refexive, transitive frames. This is both useful and limiting. On the positive side, it allows us to transfer a variety of important results directly to the topological semantics. On the negative side, most interesting topologies are non-Alexandroff (*e.g.*, metric spaces). Much of our work in what follows will be constructing "nice" maps between metric spaces and non-Alexandroff topologies.

2.4.2 Interior maps and truth preservation in the topological semantics

The work in the sections below requires us to recall some additional topological notions. In the topological semantics, the notion of an *interior map* plays the role of p-morphism in the Kripke (or frame) semantics. In fact, when the topologies in question are Alexandroff, the notions of p-morphism and interior map correspond exactly.

Let X and Y be topological spaces.

Defnition 2.4.4 (Open Map)**.** *A map $g : X \to Y$ is* open *if for every open subset $O \subseteq X$, $g(O)$ is open in Y.*

Defnition 2.4.5 (Continuous Map)**.** *A map $g : X \to Y$ is* continuous *if for every open subset $U \subseteq Y$, $g^{-1}(U)$ is open in X.*

Defnition 2.4.6 (Interior Map)**.** *A map $g : X \to Y$ is* interior *if it is both open and continuous.*

Defnition 2.4.7 (Full-Interior Map)**.** *A map $g : X \to Y$ is* full-interior *if it is interior and surjective.*

Fact 2.4.8 (Full-Interior Maps Preserve Modal Formulas)**.** *Let $g : X \to Y$ be a full-interior map, and φ any formula of the standard propositional modal language L. Let $V' : \mathsf{P} \to \mathsf{P}(Y)$ be a valuation function and let $V = ([g^{-1}] \circ V')$.*[7] *Then, for any $x \in X$,*

$$\langle X, V\rangle, x \models \varphi \quad \Leftrightarrow \quad \langle Y, V'\rangle, g(x) \models \varphi$$

Proof. The proof is by induction on the complexity of φ. The base case and the Boolean cases are straightforward. For the modal case:

$$\langle X, V\rangle, x \models \Diamond\psi \quad \Leftrightarrow \quad \langle Y, V'\rangle, g(x) \models \Diamond\psi$$

we use the preservation of open sets along g to show the left-to-right direction, and we use the continuity of g to show the right-to-left direction. The details of the proof can be found in, *e.g.*, (40). □

Now suppose that X and Y are *Alexandroff* topologies, and let F_X and F_Y be the corresponding frames. Moreover, let $g : X \to Y$ be a full-interior map. Then,

Fact 2.4.9. *The function g reinterpreted as $g : \mathsf{F}_X \to \mathsf{F}_Y$ is a p-morphism.*

Proof. See *e.g.*, (40). □

[7]Thus V is a valuation function on X, defned as the composition of g^{-1} with V'.

Just as p-morphisms play an important role in transferring completeness results in the relational semantics, interior maps play a similar role in transferring completeness results in the topological semantics. In the remainder of this section, we recall some of the better known topological completeness results for $S4$. We then use a particular sequence of interior maps to prove completeness for the Koch fractal and the real interval $[0,1]$.

2.4.3 Topological completeness results for $S4$

Theorem 2.4.10. *The logic $S4$ is sound and complete with respect to*

(i) *the class of all topologies (McKinsey & Tarski);*

(ii) *the class of all fnite topologies (Kripke);*

(iii) *any dense-in-itself metric space (McKinsey &Tarski);*

(iv) *the infnite binary tree, T_2 (see below) (van Benthem, Gabbay).*
In this chapter we will show,

(v) *a direct construction for the* Koch Curve, *K. The Minkowski-Bouligand dimension of K is 1.26. (This chapter or McKinsey &Tarski).*[8]

(vi) *the Wilson tree or complete binary tree, T^+_2, equipped with the topology generated by fnite initial segments [see Defnition 2.4.11]. (This chapter)*

Proof. (*ii*) follows from completeness for fnite frames; (*iii*) is proved in (27); (i) follows from either (*ii*) or (*iii*); (*iv*) follows from Lemma 2.3.9, originally due to van Benthem and Gabbay.[9] For (*v*) and (*vi*), see the later sections of this chapter. □

Part of our goal in this chapter is to revisit (*iii*) — in particular, the special case of the real line, R — as well as to give a direct completeness proof for the Koch curve. We will also mention some other fractals that are useful in topo-modal constructions and for which completeness results can be had. We have in mind, in particular, a direct proof of completeness of $S4$ for R^2 and R^3 via the Sierpinski Carpet and Menger Sponge, respectively.

[8]Since the standard topological dimension of K is 1, there is a homeomorphism h between K and $[0,1]$. Thus, we know that $S4$ is complete for K as we can transfer counterexamples via h. However, this is the frst direct completeness construction on a fractal curve of non-integer Minkowski-Bouligand dimension, except for Cantor Set.

[9]Both J. van Benthem and D. Gabby introduce a variant of the unravelling technique.

2.4.4 The infnite binary tree and the complete binary tree, viewed topologically

The infnite binary tree, T_2, is a rare object in mathematics that exhibits interesting structural features from a great range of different perspectives. As we saw above, it has enough structural symmetry and fexibility to carry the weight of the completeness theorem of $S4$ in the relational semantics. T_2 recurs when we start thinking of space fractally. We look next at an extension of T_2 called the Wilson tree, or complete binary tree, that allows us to prove two completeness results in the topological semantics.

Wilson tree (complete binary tree), T^+_2

Defnition 2.4.11. *Take alphabet $\Sigma = \{0, 1\}$ and construct the set $\Sigma^*(\Sigma^+)$ of all fnite (countable) strings over Σ. For any $s \in \Sigma^*$, let $B_s = \{s * t \mid t \in \Sigma^+\}$,* i.e., *the set of all (possibly infnite) strings with initial segment s (where s is allowed to be the empty string). Let $B = \{B_s \mid s \in \Sigma^*\}$. Note that B is closed under fnite intersections (For any $s, t \in \Sigma^*$ either $B_s \subseteq B_t$, $B_t \subseteq B_s$, or $B_s \cap B_t = \emptyset$), hence is a basis for some topology $\mathcal{J}^+$ over Σ^+. Finally, let $\mathsf{T}^+_2 = (\Sigma^+, \mathcal{J}^+)$.*

Fact 2.4.12. *(i) Σ^+, the underlying set of T^+_2, is uncountable;*
(ii) T^+_2 is frst countable;
(iii) T^+_2 is non-Alexandroff.
SEPARATION AXIOMS*:*
(iv) T^+_2 is T_0,
(v) T^+_2 is not T_1 (hence non-Hausdorff and non-metrizable)

Proof. (*i*) follows from an injection between the set of countably infnite strings over Σ and the real interval [0, 1]; (*ii*) follows from the fact that the basis, B, is countable; (*iii*) the intersection of basic opens B_0, B_{00}, B_{000}, ... (*i.e.*, the countable sequence $000\ldots$) is not open; (*iv*) For $s, t \in \Sigma^+$, $s \neq t$: if s is a descendant of t, then either B_s separates s and t (if $s \in \Sigma^*$) or there exists $t' \in \Sigma^*$ which is a descendant of t such that $B_{t'}$ separates s and t (and vice versa, if t is a descendant of s). If neither s nor t is a descendant of the other, there exists $t' \in \Sigma^*$ such that t' is an ancestor of s but not of t, and $B_{t'}$ separates s and t; (*v*) take, for instance, $s = 0$ and $t = 00$: there is no open set containing s that does not contain t. □

In the remainder of this section, we show that $S4$ is complete for T^+_2. To this end, recall the map $f : T_2 \to \mathsf{F} = \langle U, R, x\rangle$ given in Defnition 2.8. We view this function now as a map, $f : \Sigma^* \to U$, between underlying sets, and extend it to a map, $f^+ : \Sigma^+ U$. Moreover, we now view the frames F and T^+_2 as topological spaces, and the map f^+ as a topological map. We show that f^+ is full-interior.

Since $S4$ is complete for fnite, transitive, refexive frames, it follows from Fact 2.4.8 that $S4$ is also complete for T^{+}_{2}

We will need a few simple infnitary notions. We begin by defning an infnite branch b of the tree T_2.

Defnition 2.4.13. *[Countable Branch] Let $b = < t_0, t_1, \ldots >$ be an infnite branch of T_2. That is:*

(i) $t_0 = \langle\cdot\rangle$*;*

(ii) For each $n \in \mathbb{N}$, either $t_{n+1} = t_n * 0$ or $t_{n+1} = t_n * 1$.

Lemma 2.4.14. *[Cycling Lemma] Let f be any function from T_2 onto $\mathsf{F} = \langle U, R, x\rangle$, and let $b = < t_0, t_1, \ldots >$ be an infnite branch in T_2. Then there exists $N \in \mathbb{N}$ such that for all worlds $x \in U$, and all $m > N$,*

$$f(t_m) = x \textit{ implies } f(t_m) = x \textit{ for } \text{infnitely } \textit{many } m.$$

Proof. The lemma follows from the fact that U is fnite, so there are only fnitely many labels in U for f to "choose" from. Labels that occur only fnitely many times on a branch, occur for the last time at some fnite node of T_2. □

For a given branch b, let n_b be the least such $N \in \mathbb{N}$. Let $A_b = \{ f(t_m) : m > n_b \}$ (Thus A_b is the collection of worlds in U that label infnitely many nodes of the branch, b, under f).

Note that the Lemma states that after some initial segment of b all nodes of b are sent by f to elements in A_b and each of these elements labels infnitely many nodes on the branch.

Fact 2.4.15. *For any $n \in \mathbb{N}$ and any $x \in A_b$, $\exists m > n$ such that $f(t_m) = x$.*

Proof. This follows from the fact that every element in A_b labels infnitely many nodes in b. □

Defnition 2.4.16. *[Branch Labeling] Let f be a p-morphism from T_2 onto the fnite rooted frame $\mathsf{F} = \langle U, R, x\rangle$. For every branch b in T_2, we let the fnite choice function $C(b)$ return a choice of $y \in A_b$. Further, noting that every branch b has a unique countable sequence in Σ^* associated with it, we can think of the branches and elements of Σ^+ interchangeably. We defne the extension, $f^+ : \Sigma^+ \to U$, of f as follows: Let t_b be the element in Σ^+ that corresponds to the branch b. We let $f^+(t_b) = C(b)$. Thus we label each countable string in Σ^+ with a node in $A_b \subseteq U$.*

For the remainder of this chapter we view f^+ and f interchangeably as a maps between topological spaces, frames, or simply underlying sets. From the context it should be clear which of these we intend. Also, we refer to 'fnite' and 'limit' nodes of the tree T^+_2 with the obvious interpretation.

Theorem 2.4.17. $f^+ : \mathsf{T}^+_2 \to \mathsf{F}$ *is full-interior.*

Proof. We need to show that f^+ is open, continuous, and surjective.

(Open.) Let $O \in {}^+$ be a basic open set. Then $O = B_s$ for some fnite node s. Let $y = f^+(s)$ and let $D_y = \{z \in U \mid Ryz\}$. We show that $f^+(B_s) = D_y$. We know that every point in D_y labels some node in B_s by the fact that f is a p-morphism. Thus $D_y \subseteq f^+(B)$. For the reverse inclusion, let $z\ f^+(B_s)$. Then $z = f^+(t)$ for some $t \in B_s$. If t is fnite then $f^+(t) = f(t)\ \ D_y$, where inclusion follows from the fact that f is a p-morphism. If t is a limit node, then $f^+(t) = f^+(t')$ for some fnite node $t' \in B_s$ (by construction of f^+). Moreover, $f^+(t') = f(t') \in D_y$ (since t' is fnite). Thus $f^+(B)\ \ D_y$, as needed.

(Continuous.) Let U be an open set in F. Let $s \in (f^+)^{-1}(U)$, and let $f^+(s) = y \in U$. We need to show there is an open set $O \subseteq \mathsf{T}^+_2$, such that $s \in O$ $(f^+)^{-1}(U)$. Now if s is fnite, then we already know that $f^+(B_s) = D_y \subseteq U$ (by proof of *Open* above). So $s \in B_s\ (f^+)^{-1}(U)$, where B_s is open. If s is a limit node, then there is some fnite s' such that $f^+(s') = y$, and $Rs's$. But then $f^+(B_{s'}) = D_y\ \ U$ and $s \in B_{s'}\ \ (f^+)^{-1}(U)$, where $B_{s'}$ is open. This shows that $(f^+)^{-1}(U)$ is open, as needed.

(Surjective.) Surjectivity follows from the fact that $x \in Range(f^+)$ and f^+ is open (where x is the root of F).

□

Theorem 2.4.18. $S4$ *is complete for* T^+_2.

Proof. By Fact 2.4.8, Theorem 4.2.3, and Theorem 2.4.17. □

In the next section, we construct a full-interior function from the real interval $[0,1]$ onto T^+_2 via the Koch Curve. That construction gives us both completeness of $S4$ for the Koch Curve, and a new proof of completeness of $S4$ for the real interval $[0,1]$.

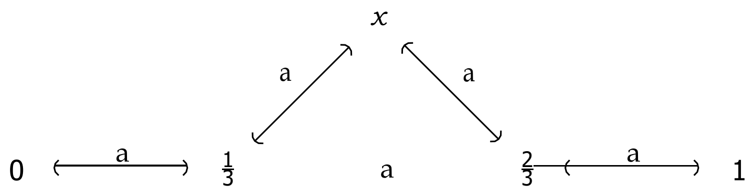

Figure 6: K_1: $a = \frac{1}{3}$

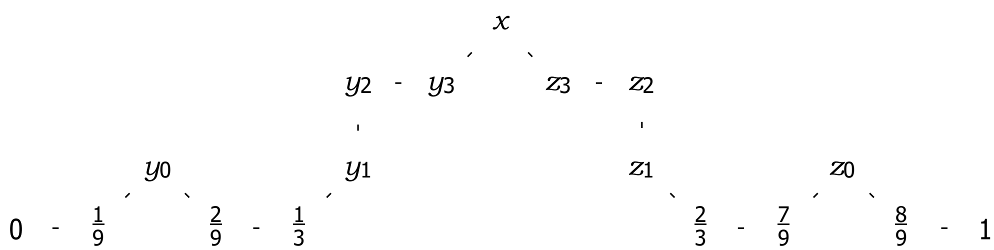

Figure 7: K_2: The length of each line segment is $\frac{1}{9}$, and the fve triangles with apex's at y_0, y_2, x, z_2, z_0 are equilateral triangles

.

2.5 Fractal curves and topological completeness

Our goal is to construct a homeomorphism between the interval $[0, 1]$ and Koch Curve fractal, K, and a relatively simple full interior labelling $l : [0, 1]$ T_2^+ inspired by the construction of Koch Curve. The labeling itself provides a straightforward proof of completeness of $S4$ for the real interval. When composed with the homeomorphism we obtain completeness of $S4$ for the singleton class K, the Koch Curve.

2.5.1 The Koch curve

Recall the construction of the Koch curve, K.

We begin with the unit interval $[0, 1]$. At the frst stage, K_1, we let the middle third of the interval be "pushed up" to form two sides of an equilateral triangle with side length $\frac{1}{3}$, as pictured in Figure 6. At the second stage we let the middle third of each line segment of K_1 be raised to form two sides of an equilateral triangle of length $\frac{1}{9}$. This gives K_2 in Figure 7.

In general, at stage n of construction, we raise the middle third of each line

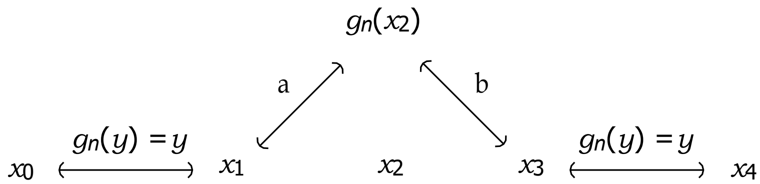

Figure 8: This fgure shows how g_n acts on a single segment $[x_0, x_4]$ of K_{n-1}. g_n is the identity function everywhere except: (*i*) $g_n(x_2)$ is the apex of the triangle, and (ii) g_n maps the line segment (x_1, x_2) linearly onto a and maps the line segment (x_2, x_3) linearly onto b.

segment of K_{n-1} to form two sides of an equilateral triangle of side length equal to the length of the segment raised.

The Koch curve is a limit of the construction stages in the following sense. Let K_0 be the unit interval $[0, 1]$. For $n = 1, 2 \ldots$, let

$$g_n : K_{n-1} \to K_n$$

be the obvious homeomorphism from K_{n-1} to K_n. And let

$$f_n = g_n \circ g_{n-1} \ldots \circ g_1$$

Thus, for each $n \in \mathbb{N}$, $f_n : [0, 1] \to K_n$ is a homeomorphism from $[0, 1]$ onto K_n. Finally, we let f be the pointwise limit of these functions:

$$f = \lim_{n \to \infty} f_n$$

and the Koch curve, K, is the range of this limit:

$$K = Range(f)$$

Claim 2.5.1. *$f : [0, 1] \to K$ is a homeomorphism.*

Proof. We need to show that f is bijective, continuous and open.

1. (Bijective) Note that any two distinct points $x, y \in [0, 1]$ eventually end up on different line segments under some f_n. Indeed, since $x = y$, we know $d(x, y) > 0$ (where d denotes the usual distance function). But the length of line segments in K_n is $(\frac{1}{3})^n$. Since $(\frac{1}{3})^n \to 0$, the length of line segments in K_n is eventually smaller than the distance $d(x, y)$, and x and y belong to

different line segments. We leave it to the reader to verify that such points are not identifed under f — *i.e.*, $f(x) = f(y)$. This shows that f is injective. Surjectivity follows from the fact that $K = Range(f)$.

2. (Continuous) We show that f is the uniform limit of continuous functions, hence continuous.[10] Note that for any $x \in [0, 1]$, $d(f_n(x), f_{n-1}(x)) = d(g_n(f_{n-1}(x)), f_{n-1}(x))$, where d denotes the distance function in the usual metric on $\mathbb{R}^2$. Moreover, by construction of g_n, g_n moves points at most a distance of $(\frac{1}{3})^n(\frac{\sqrt{3}}{2})$. So $d(f_n(x), f_{n-1}(x)) < (\frac{1}{3})^n(\frac{\sqrt{3}}{2}) \to 0$, for all x $[0, 1]$. Thus the f_n's converge uniformly, and the uniform limit of continuous functions is continuous.

3. (Open) We frst show that the image under f of a closed set is closed. Indeed, if $A \subseteq [0,1]$ is closed, then it is also compact (since [0,1] is bounded). But the continuous image of a compact set is compact, so $f(A)$ is a compact subset of K. So $f(A)$ is closed (and bounded), as desired. Now suppose that $O \subseteq [0,1]$ is open. Then $f(O) = f([0,1]) - f([0,1] - O) = K - f([0,1] - O)$, since $f : [0,1] \to K$ is a bijection. By the above argument, $f([0,1] - O)$ is closed, so $f(O)$ is open.

□

It follows from the previous claim that $f^{-1} : K \to [0,1]$ is a homeomorphism. We now wish to construct a function $l : [0,1] \to T_2^+$ that is full-interior. Once we have done so, l alone will prove completeness of $S4$ for the real interval [0,1], and the composition $l \circ f^{-1} : K \to T_2^+$ will prove completeness of $S4$ for the Koch curve, K. Much as we constructed f as a limit of fnite approximations, f_n, we now construct the function l as a limit of stagewise labeling functions, l_n. Indeed, as the reader will presently see, the functions, l_n, correspond neatly to stages of Koch construction.

Note above that each $g_n : K_{n-1} \to K_n$ sends K_{n-1} to K_n by breaking up each line segment of K_{n-1} into four line segments of K_n. For any line segment s in K_{n-1} we refer to its "successor" segments in K_n as (in order from left to right) $A(s)$, $B(s)$, $C(s)$ and $D(s)$ (see Figure 9). There is an ambiguity here with respect to endpoints: is the point $\frac{1}{3}$, for example, in the segment $A([0,1])$ or $B([0,1])$? For reasons that will become clear below, we decide that $B(s)$ and $C(s)$ are always open on both ends, while the "right" end-point of $A(s)$ and the "left" endpoint of $D(s)$ are always closed. (The left endpoint of $A(s)$ and the right endpoint of $D(s)$ are either open or closed, depending on whether the segment s itself is open or

[10] Here we view the functions f_n as functions from the space [0,1] to $\mathbb{R}^2$, with the usual metrics on each of these spaces.

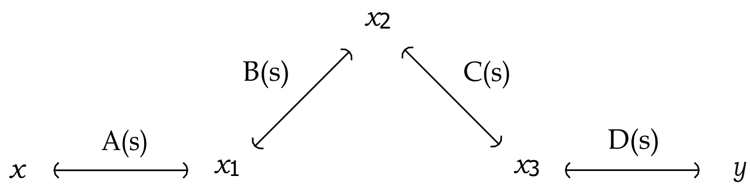

Figure 9: Segment s in K_{n-1} is $[x, y]$. Then $A(s) = [x, x_1]$, $B(s) = (x_1, x_2)$, $C(s) = (x_2, x_3)$, $D(s) = [x_3, y]$, and $E(s) = \{x_2\}$.

closed at that endpoint). Thus, *e.g.*, $\frac{1}{3} \in A([0, 1])$ and $\frac{2}{3} \in D([0, 1])$. Note that for each segment s this leaves one point still unclassifed—namely, the midpoint of s which becomes in the next stage of construction, the apex of the equilateral triangle (in Figure 9, the point x_2). For simplicity, we let this one point constitute a new singleton set $E(s)$.

These defnitions allow us to construct stages of labeling in a natural way. Fix $x \in [0, 1]$, $n \in \mathbb{N}$ and let $s_{x,n-1}$ be the line segment in K_{n-1} containing $f_{n-1}(x)$. We let:

$$l_n(x) = \begin{cases} l_{n-1}(x) * 0 & \text{if } f_n(x) \in B(s_{x,n-1}) \\ l_{n-1}(x) * 1 & \text{if } f_n(x) \in C(s_{x,n-1}) \\ l_{n-1}(x) & \textit{otherwise} \end{cases}$$

Stages of labeling correspond to stages of Koch construction. If in the n-th stage of Koch construction x "stays in the same place" (*i.e.*, $f_n(x) = f_{n-1}(x)$), then the label for x at stage n remains what it was in the previous stage (*i.e.*, $l_n(x) = l_{n-1}(x)$). If on the other hand x gets "pushed up" to a side of an equilateral triangle introduced at stage n, then the new label $l_n(x)$ appends a 0 or 1 to the old label $l_{n-1}(x)$ (depending on which side of the equilateral triangle—*i.e.*, "left" or "right".)

Note that some elements in $[0, 1]$ "stabilize" over successive labelings and some do not. More precisely, some but not all points $x \in [0, 1]$ satisfy the following condition:

$$(*) \exists N \in \mathbb{N} \text{ such that } \forall n \geq N, \; l_n(x) = l_N(x)$$

If every point in the interval stabilized, we could happily restrict our attention to the infnite binary tree T_2 (*without* limits) and use this tree to label points in the real interval [0, 1]. The fact that many—in fact uncountably many—points do *not* stabilize is our motivation for passing from T_2 to T_2^+. Our fnal labeling function, l, agrees with stage-wise labeling functions on points that stabilize, but assigns limit nodes of T_2^+ to all points that do not stabilize. We define the function $l : [0,1] \to \mathrm{T}_2^+$ as follows:

$$l(x) = \begin{cases} l_N(x) & \text{if } x \text{ satisfies } (*) \\ t & \text{otherwise} \end{cases}$$

where t is the unique countable sequence over $\{0,1\}$ that has $l_n(x)$ as initial segment for each $n \in \mathbb{N}$.

To take a simple example, it is clear that the point $\frac{1}{3}$ stabilizes and therefore $l(\frac{1}{3})$ is a finite string. Indeed, $l(\frac{1}{3}) = \langle\rangle$, as $l_n(\frac{1}{3}) = l_0(\frac{1}{3}) = \langle\rangle$ for all n $\mathbb{N}$. Note that successive labeling functions, l_n, are monotonic in the following sense: For any $x \in [0,1]$, if $m < n$, then $l_n(x)$ is an descendant of $l_n(x)$ (*i.e.*, $l_n(x) = l_m(x) * t$ for some $t \in \Sigma^+$). Moreover, $l(x)$ is a descendant of $l_n(x)$ for all $n \in \mathbb{N}$ (*i.e.*, $l(x) = l_n(x) * t_n$ for some $t_n \in \Sigma^+$).

Theorem 2.5.2. $l : [0,1] \to \mathrm{T}_2^+$ *is a full, interior map*

The proof of this theorem is given in the section below. We state as corollaries the two main results of this chapter:

Corollary 2.5.3. *S4 is complete for the class of models over the real interval* $[0,1]$.

Proof. Immediate from Fact 2.4.8, Theorem 2.4.18, and Theorem 2.5.2. □

Corollary 2.5.4. *S4 is complete for the class of models over Koch curve, K.*

Proof. By the map $l \circ f^{-1} : K \to \mathrm{T}_2^+$. That the composition is full-interior is immediate from Claim 2.5.1 and Corollary 2.5.3. □

2.5.2 Completeness via the Koch curve

In this section, we prove Theorem 2.5.2.

Proof. As before, for any finite node $s \in \mathrm{T}_2^+$, let B_s be the basic open set $\{s * t \mid t \in \Sigma^+\}$.

1. (Continuous) Let U be a basic open set in T_2^+. Then $U = B_s$ for some finite node $s \in \mathrm{T}_2^+$. Suppose $x \in l^{-1}(B_s)$. We show there is an open set $O \subseteq [0,1]$ such that $x \in O \subseteq l^{-1}(B_s)$. By construction of the functions l_n, there exists a least $N \in \mathbb{N}$ such that $l_N(x) = s$. Moreover, at stage N all points belonging to some open interval O which contains x are labeled by s—*i.e.*, for each $y \in O$, $l_N(y) = s$. By monotonicity of the labeling functions, $l(y)$ is a descendant of $l_N(y) (= s)$ for each $y \in O$. So $O \subseteq l^{-1}(B_s)$. Moreover, $x \in O$ and O is open, as needed.

2. (Open) We introduce the notion of a maximal, uniformly labeled (MUL) interval under l_n. In particular, $I \subseteq [0, 1]$ is a MUL interval under l_n if for all $x, y \in I$, $l_n(x) = l_n(y)$, and there does not exist a strictly bigger interval $I^0 \supset I$ with this property. With slight abuse of notation, where I is a MUL interval under l_n, all of whose points are labeled by some node t, we will write $l_n(I) = t$. Note that for each point $x \in [0, 1]$, x belongs to successively smaller MUL intervals under the fnite labeling functions, $l_1, l_2, l_3, \ldots$. (Thus, *e.g.*, for $x = 1/4$, x belongs to the MUL interval $[0, \frac{1}{3}]$ under l_1, then to the MUL interval $[\frac{2}{9}, \frac{1}{3}]$ under l_2, *etc.*) Letting $I_{x,n}$ be the MUL interval under l_n containing x, we have that $length\,(I_{x,n}) \to_{n\to\infty} 0$. It follows that if $O \subseteq [0, 1]$ is open, and $x \in O$, then for large enough n, $I_{x,n} \subseteq O$.

Now let $O \subseteq [0, 1]$ be open, and suppose $s \in l(O)$—that is, $l(x) = s$ for some $x \in O$. We need to show that there exists an open set $U \subseteq T_2^+$ such that $s \in U \subseteq l(O)$.

If (case 1) s is fnite, then for large enough n, $I_{x,n} \subseteq O$ and $l_n(I_{x,n}) = s$. We claim that $l(I_{x,n}) = B_s$. Since $I_{x,n} \subseteq O$, we have $s \in B_s \subseteq l(O)$, and B_s is open, as needed. (Proof of the claim: By monotonicity of the labeling functions, we know that $l(I_{x,n}) \subseteq B_s$. The diffcult part is to show that $B_s \subseteq l(I_{x,n})$—in particular, that every *limit node* in B_s labels some point in $I_{x,n}$ under l. We prove this part, and leave the case for fnite nodes to the reader. Let r be a limit node in B_s. Then $r = s * r^0$ for some countably infnite string $r^0 \in \Sigma^+$. We write $r^0 = (r^0_1, r^0_2, r^0_3, \ldots)$. We need to fnd $x^0 \in I_{x,n}$ such that $l(x^0) = r$. It will be useful for us to label different segments of an MUL interval, I, by $A(I)$, $B(I)$, $C(I)$, and $D(I)$, just as we labeled different parts of the line segments in K_n above.[11] We now defne a sequence of points $x_n \in [0, 1]$, recursively. For the base step: If $r^0_1 = 0$, then let x_1 be some point in $B(I_{x,n})$; if $r^0_1 = 1$, then let x_1 be some point in $C(I_{x,n})$. For the recursive step, assume we have defned the points $x_1, \ldots, x_k$. Then if $r^0_{k+1} = 0$, let x_{k+1} be some point in $B(I_{x_k,n+k})$; if $r^0_{k+1} = 1$, then let x_{k+1} be some point in $C(I_{x_k,n+k})$. By construction, for each $k \in \mathbb{N}$, we have $x_{k+1}, x_k \in I_{x_k,n+k}$. So $|x_{k+1} - x_k| \le length\,(I_{x_k,n+k}) \to_{k\to\infty} 0$. Thus the sequence $\{x_k\}$ is Cauchy, hence convergent. We let $x^0 = \lim_{k\to\infty} x_k$.

[11] Thus, if $I = (i_1, i_2)$, we have:

$$B(I) = (i_1 + \tfrac{1}{3}(i_2 - i_1),\ i_1 + \tfrac{i_2 - i_1}{2})$$

$$C(I) = (i_1 + \tfrac{i_2 - i_1}{2},\ i_2 - \tfrac{1}{3}(i_2 - i_1))$$

It is then clear by construction that $x' \in I_{x,n}$ and $l(x') = s * r' = r$, as needed.)

If (case 2) s is a limit node, then $l_n(I_{x,n})$ is a fnite ancestor of s, for each $n \in \mathsf{N}$. We pick n large enough so that $I_{x,n}$ O and let $t = l_n(I_{x,n})$. Then, as in the previous case, $l(I_{x,n}) = B_t$. Moreover, $s \in B_t$ by monotonicity of the labeling functions. Since $I_{x,n} \subseteq O$, we have $s \in B_t \subseteq l(O)$, and B_t is open, as needed.

3. (Surjective) We know already that for some $x \in [0, 1]$, $l(x) =$, which is the root of T_2^+ (pick, *e.g.*, $x = \frac{1}{3}$). Moreover, the entire interval $[0, 1]$ is open. So by the fact that l is open, $l[0, 1]$ is open, and contains the root of T_2^+. Since every node in T_2^+ is a descendant of the root, it follows that l is surjective.

□

This completes the proof of the theorem.

Chapter 3

Completeness of $S4$ for the Lebesgue Measure Algebra

This chapter explores a new, probabilistic semantics for the basic propositional modal language. In a series of recent talks, Dana Scott showed that the standard propositional modal language can be interpreted probabilistically, by assigning formulas to elements of the *Lebesgue measure algebra*, or algebra of Borel subsets of [0,1] modulo sets of measure zero. In this semantics, formulas are not simply true or false in a given model, but acquire a probability value between 0 and 1, corresponding to the measure of the element of the algebra to which they are assigned. We prove completeness of $S4$ for Scott's semantics (formally, that $S4$ is complete for the Lebesgue measure algebra). Several interesting corollaries follow from the proof of this result. First, any non-theorem of $S4$ can be refuted at each point in a subset of the real interval $[0, 1]$ of measure arbitrarily close to 1. Second, intuitionistic propositional logic (IPC) is complete for the subframe of *open* elements in the Lebesgue measure algebra, or elements that have an open representative.

3.1 Introduction

[1] We saw, in the previous chapter, that modal languages can be interpreted in topological spaces, and that the modal logic $S4$ characterizes any dense-in-itself metric space — in particular, the real line, $\mathbb{R}$. The real line, however, can be investigated not just from a topological point of view, but from a measure-theoretic point of view. Here, the probability measure we have in mind is the usual Lebesgue measure on the reals. In the last several years Dana Scott introduced a new *probabilistic* or *measure-based* semantics for $S4$, which is built around Lebesgue measure on the reals.

Scott's semantics is essentially algebraic: formulas are interpreted in the *Lebesgue measure algebra*, or the σ-algebra of Borel subsets of the real interval [0,1], modulo sets of measure zero (henceforth, "null sets"). We denote this algebra by $\mathcal{M}$. Thus elements of $\mathcal{M}$ are equivalence classes of Borel sets. In Scott's semantics, each propositional variable is assigned to an arbitrary element of $\mathcal{M}$. Conjunctions, disjunctions and negations are interpreted as meets, joins and complements in the algebra, respectively. In order to interpret the $S4$ '$\Box$'-modality, we add to the algebra an *interior* operator (defned below), which we construct from the collection of *open* elements in the algebra, or elements that have an open representative. Unlike the Kripke or topological semantics, there is no notion here of truth *at a point* (or at a "world"). Indeed, singleton sets — sets consisting of a single point — have measure zero, and so "disappear" in the Lebesgue measure algebra.

The introduction of a new semantics brings with it familiar questions. Is the set of validities in the Lebesgue measure algebra axiomatizable? If so, is it characterized by any known modal logic? In particular, does the set of validities in the measure algebra coincide with the theorems of $S4$ (*i.e.*, is $S4$ sound and complete for Scott's measure-based semantics)? Such questions belong to a broader family of questions that parallel, in some sense, the questions that we are accustomed to ask about Tarski's topological semantics. Do different measure algebras give rise to different modal logics? To what extent can modal languages describe, discriminate between, and help us to reason about different measure structures?

In this chapter, we address the question of completeness for Scott's semantics. Our main result is that $S4$ is complete for the Lebesgue measure algebra. Two important corollaries follow from the proof of this result. First, any non-theorem of $S4$ can be refuted at each point in a subset of the real interval, [0, 1], of measure

[1] A version of this paper was published in ***Journal of Philosophical Logic*** (see (22)). Since then I thought of an easier way to go about the main proof, and so parts of the current version are changed from the published version. This easier way is inspired by the main construction in (38). I would like to thank the publishers of the ***Journal of Philosophical Logic*** for granting me the permission to reproduce the published work here.

arbitrarily close to 1. Second, intuitionistic propositional logic (IPC) is complete for the subframe of 'open' elements in the Lebesgue measure algebra, or elements with an open representative set.

3.2 Topological and algebraic semantics for $S4$

Let the propositional modal language L consist of a countable set, $\mathsf{P} = \{P_i \mid$ for all $i \in \mathsf{N}\}$, of propositional variables and be closed under binary connectives $\rightarrow, \vee, \wedge, \leftrightarrow$ and unary operators $\neg$, D, *3*.

Defnition 3.2.1. *The modal logic $S4$ in the language L consists of some complete axiomatization of classical propositional logic PL, some complete axiomatization of the minimal normal modal logic K, say the axiom:*

$K : \mathrm{D}(\varphi \rightarrow \psi) \rightarrow (\mathrm{D}\varphi \rightarrow \mathrm{D}\psi)$

and the rule:

$N :$ \` $\varphi \Rightarrow$ \` $\mathrm{D}\varphi$

and fnally the two special $S4$ axioms:

$4 : \mathrm{D}P \rightarrow \mathrm{DD}P$
$T : \mathrm{D}P \rightarrow P$

We are interested in algebraic models of the modal system $S4$, or topological Boolean algebras.

Defnition 3.2.2. *A* topological Boolean algebra *(henceforth TBA) is a Boolean algebra with an* interior *operator, I, satisfying the following properties:*

(I_1) $Ia \leq a$
(I_2) $I(a \wedge b) = Ia \wedge Ib$
(I_3) $IIa = Ia$
(I_4) $I(1) = 1$

A *complete* TBA is a TBA in which every collection of elements has a supremum (and infmum).

Example 3.2.3. *(Topological feld of sets) The set of subsets $\mathsf{P}(X)$ of a topological space X with set-theoretic meets, joins and complements, and where Ia denotes the (topological) interior of a, is a complete TBA and we denote it by $B(X)$. More generally, any Boolean algebra, A of subsets of a topological space X that is closed under topological interiors is a TBA (A need not contain all subsets of X). We call any such algebra a* topological feld of sets. *Note that we reserve the notation $B(X)$ for the topological Boolean algebra generated by* all *subsets of X.*

Defnition 3.2.4. *An* algebraic model *of* $S4$ *is a pair,* $\langle \mathbb{A}, V \rangle$, *where* $\mathbb{A}$ *is a topological Boolean algebra, and* $V : \mathbb{P} \to \mathbb{A}$ *is a valuation function, assigning to each propositional variable some element of the algebra,* $\mathbb{A}$.

We would like to extend the valuation function, V, to the set of all formulas in L, and we do so by the following recursive clauses. For any formulas φ and ψ, let:

$$\begin{aligned} V(\varphi \vee \psi) &= V(\varphi) \vee V(\psi) \\ V(\neg\varphi) &= -V(\varphi) \\ V(\Box\varphi) &= I(V(\varphi)) \end{aligned} \tag{3.1}$$

where symbols on the RHS denote (in order) the algebraic join, complement, and interior. (The remaining binary connectives $\{\&, \to, \leftrightarrow\}$ and unary operator $\{\Diamond\}$ are defned in terms of the above in the usual way.)

Let $M = \langle \mathbb{A}, V \rangle$ be an algebraic model. We say a formula φ is *satisfed* in M ($M \models \varphi$) iff $V(\varphi) = 1_{\mathbb{A}}$ (the top element in the algebra). We say φ is *satisfed in* $\mathbb{A}$ ($\mathbb{A} \models \varphi$) iff φ is satisfed in every model M defned over the algebra $\mathbb{A}$. Finally, for any class C of TBA's, φ is *satisfed in* C ($\models_C \varphi$) iff φ is satisfed in every TBA in C.[2]

We now defne completeness in the usual way: A logic S is *complete* for a class, C, of TBA's if every formula that is satisfed in C is provable in S. In symbols,

$$\models_C \varphi \;\Rightarrow\; \vdash_S \varphi$$

An equivalent formulation will be more useful in what follows: S is complete for C if any non-theorem of S is refuted in C. In symbols,

$$\nvdash_S \varphi \;\Rightarrow\; \not\models_C \varphi$$

Note that if $\mathbb{A}$ is a topological feld of sets, it makes sense to talk about truth *at a point* (much like truth at a world in Kripke semantics for the standard propositional modal language). For any formula φ, valuation $V : \mathbb{P} \to B(X)$, and point $x \in X$, we can say that φ is true at x if

$$x \in V(\varphi)$$

This ternary relation between a valuation, formula and point in the topological space has no place in the general algebraic semantics—where $\mathbb{A}$ need not be a topological feld of sets—and, in particular, has no analog when it comes to the Lebesgue measure algebra, as we will see below.

[2] This semantics can be generalized by defning a set of designated elements, $D_{\mathbb{A}}$, of $\mathbb{A}$ and letting satisfaction in a model $M = \langle \mathbb{A}, V \rangle$ be defned by: $V(\varphi) \in D_{\mathbb{A}}$. The defnition used in this chapter is the special case where $D_{\mathbb{A}} = \{1_{\mathbb{A}}\}$.

Theorem 3.2.5 (Tarski's completness theorem)**.** *The modal logic $S4$ is sound and complete for:*

(i) *The class of all topological spaces (*i.e., $\{B(X) \mid X$ *is a topological space* $\}$*).*

(ii) *The class of all fnite topological spaces (*i.e., $\{B(X) \mid X$ *is a fnite topological space* $\}$*).*

(iii) *Any dense-in-itself metric space (*i.e., $B(X)$ *for any dense-in-itself metric space,* X*).*

Proof. The theorem was proved by McKinsey and Tarski in 1944 in (27). □

Defnition 3.2.6. *A $S4$* Kripke frame *is a pair $\langle U, R\rangle$, where U is a set (of 'worlds') and R is a refexive, transitive binary relation on U. A* rooted *$S4$* Kripke frame *is a triple $\langle U, R, w_0\rangle$, where U and R are as above, $w_0 \in U$, and $w_0 R w$ for each $w \in U$. We say that a (rooted) Kripke frame is* fnite *if U is a fnite set.*[3]

Defnition 3.2.7. *Let X be a topological space. Then X is* Alexandroff *if the collection of open sets in X is closed under arbitrary intersections.*

It is well-known that $S4$ Kripke frames are just Alexandroff spaces, and vice versa. Indeed, let $\langle U, R\rangle$ be a $S4$ Kripke frame, and say that a set $U' \subseteq U$ is *open* if it is closed under the binary relation R. The collection of open sets so defned contains the empty set, the entire space U, and is closed under arbitrary unions and intersections. Thus the collection of open sets defnes a topology on U. Conversely, if X is an Alexandroff space, then for any $x \in X$, the set $U_x = \bigcap\{O \text{ open} \mid x \in O\}$ is an open set. We put xRy iff $y \in U_x$. The reader can verify that R is refexive and transitive. It follows that $\langle X, R\rangle$ is a $S4$ Kripke frame.

Notice that any fnite topology is Alexandroff. (There are only fnitely many points in the space, so only fnitely many open subsets.) Thus the collection of fnite topological spaces is just the collection of fnite $S4$ Kripke frames. We can now state Theorem 3.2.5 (*ii*) as follows: $S4$ is complete for the class of all fnite $S4$ Kripke frames. In fact, more is true: $S4$ is complete for the class of all *rooted* fnite Kripke frames. That is to say, any non-theorem, α, of $S4$ can be refuted at the root of a fnite Kripke frame. (We do not reprove this classic result here. To understand it, though, think about what happens if we simply delete from a (non-rooted) Kripke frame every node not related under R to the world at which α is refuted.) In the fnal section of this chapter, we will appeal to this stronger completeness result.

[3]This somewhat non-standard defnition of Kripke frames is meant to highlight frames as topological spaces. On a more standard presentation, a Kripke frame is what I call here a *rooted* Kripke frame.

3.3 The Lebesgue measure algebra

In this section we define our central object of study: the measure algebra, M. We prove that M is a complete Boolean algebra, and define an *open* sublattice in M. We then show that the sublattice of open elements forms a complete Heyting algebra.

Definition 3.3.1. *Let* A *be a Boolean algebra. We say that a non-empty subset* $I \subseteq \mathsf{A}$ *is an* ideal *if*

1. *For all* $a, b \in I$, $a \vee b \in I$
2. *If* $a \in I$ *and* $b \leq a$, *then* $b \in I$

If I *is closed under countable suprema, we say* I *is a* σ-ideal.

We can construct new Boolean algebras from existing ones by quotienting by an ideal. If A is a Boolean algebra and $I \subseteq \mathsf{A}$ is an ideal, we define the correspondence $\sim$ on A by:

$$x \sim y \text{ iff } (x \triangle y) \in I$$

where $\triangle$ denotes symmetric difference.[4] Letting A/I be the set of equivalence classes under $\sim$, and letting $|x|$ be the equivalence class corresponding to $x \in \mathsf{A}$, the operations $\vee$, $\wedge$ and $\neg$ on A/I are defined in the obvious way:

$$\begin{aligned} |x| \vee |y| &= |x \vee y| \\ |x| \wedge |y| &= |x \wedge y| \\ \neg|x| &= |\neg x| \end{aligned} \tag{3.2}$$

It is easy to verify that A/I is a Boolean algebra with top and bottom elements $|1_{\mathsf{A}}|$ and $|0_{\mathsf{A}}|$, respectively. From the definitions of $\vee$ and $\wedge$ we can reconstruct the lattice order $\leq$ as follows. For any $|x|, |y| \in \mathsf{A}/I$,

$$|x| \leq |y| \text{ iff } |x| \wedge |y| = |x|$$

Lemma 3.3.2. *Let* A *be a Boolean Algebra and* I *an ideal in* A. *Then for any elements* a, b *in the quotient algebra* A/I, *the following are equivalent:*

(i) $a \leq b$

[4] Note that differences and symmetric differences are defined in any Boolean algebra, not just in fields of sets. In particular, $x - y$ is defined as $x \wedge \neg y$ (where $\neg y$ is the Boolean complement of y) and $x \triangle y$ is defined as $(x - y) \vee (y - x)$.

(ii) *For any representatives A of a, and B of b, there exists some $N \in I$ with $A \leq B \vee N$ (in the Boolean algebra, A).*

(iii) *For any representative A of a, there exists a representative B of b with $A \leq B$ (in the Boolean algebra, A).*

Proof. $(i) \to (ii)$ Suppose $a \leq b$ and let $a = |A|$, $b = |B|$. Then $|A \wedge B| = |A| \wedge |B| = |A|$, so $A \wedge B \sim A$. Thus $A \triangle B = A \triangle (A \wedge B) = N$ for some $N \in I$. It follows that $A \leq B \vee N$. $(ii) \to (iii)$ This follows from the fact that $B \vee N \sim B$ for $N \in I$. $(iii) \to (i)$. If $A \leq B$, then $|A| \wedge |B| = |A \wedge B| = |A|$, and $a = |A| \leq |B| = b$. □

We want to add measure-structure to Boolean algebras. The simplest such structures are Boolean algebras carrying a fnitely additive measure. We are interested, however, in Boolean σ-algebras carrying a countably additive measure. The relevant defnition is given below.

Defnition 3.3.3. *A* measure, *μ, on a Boolean σ-algebra*[5], A, *is a real-valued, non-negative function μ on A, with $\mu(0_{\mathsf{A}}) = 0$, that satisfes countable additivity: If $\{F_n\}_{n \in \mathsf{N}}$ is a countable collection of elements in A with $F_n \wedge F_m = 0_{\mathsf{A}}$ for all $n, m \in \mathsf{N}$, then*

$$\mu\Big(\bigvee_{n \in \mathsf{N}} F_n\Big) = \sum_{n \in \mathsf{N}} \mu(F_n)$$

We say that a measure, μ, on a Boolean σ-algebra, A, is *normalized* if $\mu(1_{\mathsf{A}}) = 1$. We say that μ is *positive* if $\mu(a) = 0$ iff $a = 0_{\mathsf{A}}$.

Defnition 3.3.4. *(Halmos) A measure algebra is a Boolean σ-algebra, A, together with a positive, normalized measure, μ, on A.*

Fact 3.3.5. *Let μ be a normalized measure on a Boolean σ-algebra, A, and let U be the set of elements $a \in \mathsf{A}$ with $\mu(a) = 0$. Then,*

(i) *U is a σ-ideal in A*

(ii) *The quotient A/U is a Boolean σ-algebra.*

(iii) *There exists a unique measure ν on A/U defned by*

$$\nu(|a|) = \mu(a)$$

Moreover, ν is positive and normalized.

[5]A Boolean σ-algebra is a Boolean algebra that is closed under countable joins (and meets).

Proof. (*i*) If $a \leq b$, and $\mu(b) = 0$, we write $b = a \vee (b - a)$. But then $\mu(a) \leq \mu(b)$, by additivity of μ, so $\mu(a) = 0$. If $\{a_n \mid n \in \mathbb{N}\}$ is a countable collection of elements in $\mathbb{A}$ with $\mu(a_n) = 0$ for all $n \in \mathbb{N}$, then by countable subadditivity of μ, $\mu(\bigvee_n a_n) \leq \sum_n \mu(a_n) = 0$. (*ii*) We need to show that the quotient algebra $\mathbb{A}/U$ is closed under countable joins. Let $\{a_n \mid n \in \mathbb{N}\}$ be a collection of elements in $\mathbb{A}/U$, with $a_n = |A_n|$ for each $n \in \mathbb{N}$. We claim $\bigvee_n a_n = |\bigvee_n A_n|$. Clearly $|\bigvee_n A_n|$ is an upper bound on $\{a_n \mid n \in \mathbb{N}\}$. If $b = |B|$ is an upper bound on $\{a_n \mid n \in \mathbb{N}\}$, then $|A_n| = a_n \leq |B|$, and $A_n \leq B \vee N_n$ for some $N_n \in U$ (see Lemma 3.3.2). But then $\bigvee_n A_n \leq B \vee \bigvee_n N_n$, and $\bigvee_n N_n \in U$ (since U is a σ-ideal). So $|\bigvee_n A_n| \leq |B| = b$. (*iii*) the proof can be found in, *e.g.*, (15). □

Let $Leb([0,1])$ be the σ-algebra of Lebesgue-measurable subsets of the real interval $[0,1]$, and let μ denote standard Lebesgue measure. Then μ is a normalized measure on $Leb([0,1])$ with $\mu(\emptyset) = 0$.

Defnition 3.3.6. ***(The Lebesgue Measure Algebra,*** $\mathbb{M}$***)*** *Let $Null_\mu$ be the set of measure zero subsets of $[0,1]$. Then by Fact 3.3.5, the quotient algebra,*

$$Leb([0,1])/Null_\mu$$

is a measure algebra. We denote this algebra by $\mathbb{M}$ and refer to it as the Lebesgue measure algebra.

In what follows, we use uppercase letters $A, B, C\ldots$ to denote subsets of $[0,1]$ and lower-case letters $a, b, c\ldots$ to denote elements of $\mathbb{M}$. Equivalence classes of measurable sets are denoted with a bar above the relevant set (*e.g.*, $a = \overline{A}$, $0_{\mathbb{M}} = \overline{\emptyset}$, $1_{\mathbb{M}} = \overline{[0,1]}$). We use 'measure (A)' or simply '$m(A)$' to denote the measure of the set A. The defnitions in (3.2) give, for any subsets A and B of [0,1]:

$$\begin{aligned} \overline{A} \vee \overline{B} &= \overline{A \cup B} \\ \overline{A} \wedge \overline{B} &= \overline{A \cap B} \\ -\overline{A} &= \overline{[0,1] - A} \end{aligned} \tag{3.3}$$

Lemma 3.3.7. *For any sets $A, B \in Leb([0,1])$,*

$$A \sim B \quad \textit{iff} \quad \overline{A} \leq \overline{B} \textit{ and } m(A) = m(B)$$

Proof. The left-to-right direction is obvious. For the right-to-left direction, suppose $\overline{A} \leq \overline{B}$ and $m(A) = m(B)$. Then $A \subseteq B \cup N$ for some $N \in Null$, so $m(A - B) = 0$. Furthermore,

$$m(B - A) = m(B) - m(B \cap A) = m(A) - m(B \cap A) = m(A - B)$$

and we have $m(B - A) = 0$. Thus $A \triangle B \in Null$ and $A \sim B$. □

Proposition 3.3.8. M *is a complete Boolean algebra*

Proof. [6] We show that any well-ordered subset S of M has a least upper bound. The proof is by transfinite induction on the order type of S. Let S have order type α and write $S = \{p_\gamma \mid \gamma < \alpha\}$. For $\beta < \alpha$, let $q_\beta = sup\{p_\gamma \mid \gamma < \beta\}$ (existence follows from the inductive hypothesis). If α is a limit ordinal then $\{q_\beta \mid \beta < \alpha\}$ is a non-decreasing sequence of elements in M and $\{m(q_\beta) \mid \beta < \alpha\}$ is a non-decreasing sequence of reals. But note that there are only *countably* many distinct reals in this sequence (for each "jump" between two reals in the sequence, there is a distinct rational number.) It follows from Lemma 3.3.7 that there are only *countably* many distinct elements 'q_β' in the sequence $\{q_\beta \mid \beta < \alpha\}$. But M is closed under countable suprema (see Fact 3.3.5 (ii)), so $sup\ S = sup\{q_\beta \mid \beta < \alpha\}$ exists. □

By contrast, $Leb([0,1])$ is *not* a complete Boolean algebra. If, *e.g.*, S is a non-measurable subset of [0, 1], then the collection $\{\{x\} \mid x \in S\}$ has no supremum in $Leb([0,1])$. Note that the Lebesgue measure, μ, on $Leb([0,1])$ is *not* a positive measure: any non-empty countable set has measure zero, but is not equal to the bottom element, $\emptyset$ of the algebra. Indeed, it is proved in (15) that every (positive, normalized) measure algebra is complete.

The Lebesgue measure algebra M is well-known, but now we would like to turn M into a topological Boolean algebra. To do so, we must define an interior operator on the algebra. We do this by first defining a collection of 'open' elements in M.

Definition 3.3.9. *We say an element* $a \in \mathsf{M}$ *is* open *if some representative A of a is an open subset of* $[0,1]$. *We denote the set of open elements in* M *by* G.

The next proposition states that not all elements of M are open.

Proposition 3.3.10. $\mathsf{M} \neq \mathsf{G}$

Proof. The proof is postponed until §3.5.1, where we introduce thick Cantor sets. □

In the next proposition we show that open elements in M form a complete Heyting algebra. Recall that a complete Heyting algebra is a complete lattice that satisfies the following infinite distributive law: For any $x \in A$ and $\{a_i \mid i \in I\} \subseteq A$,

[6] This proof was suggested to me by Dana Scott. In fact, the more general claim that every (positive, normalized) measure algebra is complete is proved in (15). The proof procedes by showing that an algebra is complete iff it satisfies the countable chain condition, and that any measure algebra so defined satisfies this condition.

$$x \wedge \bigvee_{i\in I} a_i = \bigvee_{i\in I}(x \wedge a_i) \tag{3.4}$$

Proposition 3.3.11. G *is a complete Heyting algebra.*[7]

Proof. We need to show that G is a complete lattice. Let $\{a_i \mid i \in I\} \subseteq \mathsf{G}$, and let $a_i = \overline{A_i}$ for each $i \in I$, with A_i an open representative of a_i. Let $\{(p_n, q_n) \mid n \in \mathsf{N}\}$ be the collection of open rational intervals (open intervals with rational endpoints) contained in some (or other) A_i. We claim that $\bigvee_i a_i = \overline{\bigcup_n (p_n, q_n)}$. Clearly RHS is an upper bound on $\{a_i \mid i \in I\}$ (this follows from the fact that each open set, A_i is equal to the union of rational intervals contained in it). Suppose $b = \overline{B}$ is an upper bound on $\{a_i \mid i \in I\}$ with $b \in \mathsf{G}$.[8] For each $i \in I$, choose $N_i \in Null$ such that $A_i \subseteq B \cup N_i$. For each $n \in \mathsf{N}$, choose $i(n)$ such that $(p_n, q_n) \subseteq A_{i(n)}$. We have: $\bigcup_n (p_n, q_n) \subseteq \bigcup_n A_{i(n)} \subseteq B \cup \bigcup_n N_{i(n)}$, where $\bigcup_n N_{i(n)} \in Null$.

So $\overline{\bigcup_n (p_n, q_n)} \leq \overline{B} = b$, proving the claim. This shows that every collection of elements in G has a supremum. What about infma? Consider now the collection of $\{b_j \mid j \in J\}$ of lower bounds[9] in G on $\{a_i \mid i \in I\}$. This collection has a supremum, b. We claim that $b = \bigwedge_i a_i$. The proof is similar to the previous and is left to the reader.

Note that the proof shows that $\bigvee_i a_i = \overline{\bigcup_i A_i}$, where A_i is any open representative of a_i (for $i \in I$). We use this fact to show that satisfes the distributive law (3.4), as follows. Let $x = \overline{X}$, with X an open representative. Then,

$$\begin{aligned}
x \wedge \bigvee_i a_i &= \overline{X} \wedge \overline{\bigcup_i A_i} \\
&= \overline{X \cap \bigcup_i A_i} \\
&= \overline{\bigcup_i (X \cap A_i)} \\
&= \bigvee_i \overline{(X \cap A_i)} \\
&= \bigvee_i (x \wedge a_i)
\end{aligned}$$

[7] In general, infma in G and M do not coincide. Example: For each $n \in \mathsf{N}$, let K_n denote the set of points belonging to "remaining intervals" at the n-th stage of construction of K (defned in §3.5.1). Then $\overline{K_n} \in \mathsf{G}$ for each $n \in \mathsf{N}$, but $inf_{\mathsf{M}}\{\overline{K_n} \mid n \in \mathsf{N}\} = \overline{K}$, and $inf_{\mathsf{G}}\{\overline{K_n} \mid n \in \mathsf{N}\} = \overline{\emptyset}$ (where inf_{M} and inf_{G} denote infma in M and G, respectively).

[8] The reader can verify that the condition $b \in \mathsf{G}$ does no work in the proof. Indeed, this shows that suprema in M and G coincide. This is not the case for infma (see note. 6).

[9] It is crucial that we take lower bounds in G and not in the larger M. In general, the set of lower bounds in G and M do not coincide! See note 6.

□

With our defnition of open elements in hand, we can now equip M with an interior operator I defned as follows. For any $a \in \mathsf{M}$,

$$Ia = sup\{b \text{ open} \mid b \leq a\} \tag{3.5}$$

Proposition 3.3.12. *I is an interior operator.*

Proof. Let $a, b \in \mathsf{M}$. Axiom (I_1) is obvious. For (I_2), note that $I(a \wedge b) \leq I(a)$ and $I(a \wedge b) \leq I(b)$. So $I(a \wedge b) \leq Ia \wedge Ib$. For the reverse inequality, note that $Ia \leq a$ and $Ib \leq b$. Thus $Ia \wedge Ib \leq a \wedge b$. Moreover, $(Ia \wedge Ib) \in \mathsf{G}$. It follows that $Ia \wedge Ib \leq sup\{c \in \mathsf{G} \mid c \leq a \wedge b\} = I(a \wedge b)$. For (I_3) note that $Ia \in \mathsf{G}$, and $Ia \leq Ia$, giving $Ia \leq sup\{c \in \mathsf{G} \mid c \leq Ia\}$. By (I_1) we also have $IIa \leq Ia$. Finally for (I_4), note that $[0,1] \in \mathsf{G}$. Thus $I[0,1] = sup\{c \in \mathsf{G} \mid c \leq [0,1]\} = [0,1]$. □

Remark 3.3.13. *At this point, the reader may be wondering: Why not defne the operator I via the topological interior on underlying sets (just as Boolean operations on M are defned via set-theoretic operations on underlying sets):*

$$I(\overline{A}) = \overline{Int(A)} \tag{*}$$

*(where '$Int(A)$' denotes the topological interior of the set $A \subseteq [0,1]$). A simple example shows that defnition (*) is not correct (*i.e.*, not well-defned). Let $A = [0,1] \setminus \mathbb{Q}$. Then $A \sim [0,1]$. But $Int(A) = \emptyset$, and $Int([0,1]) = [0,1]$. So according to (*), $[0,1] = I(A) = \emptyset$.[10])*

Corollary 3.3.14. *The Measure Algebra, M, with unary operator I is a TBA.*

Proof. Immediate from Proposition 4.8 and Proposition 3.3.12. □

In general, there is no easy way to calculate the supremum of an uncountable collection of elements in M as indicated by the non-constructive proof of Proposition 4.8. However, when we calculate Ia, we take the supremum of a collection of *open* elements, and arbitrary joins of open elements reduce to countable joins, and so are well-behaved (see proof of Proposition 3.3.11). The following proposition shows how to calculate the interior operator in M in terms of underlying sets.

[10] Indeed, the example shows that the interior operator in the topological felds of sets Leb([0,1]) and $B([0,1])$ behaves quite differently from the interior operator in M. This is crucial in what follows, where, despite this difference, we aim to transfer valuations over $B([0,1])$ to M.

Proposition 3.3.15. *Let $a \in \mathsf{M}$ and let $\{(p_n, q_n) \mid n \in \mathsf{N}\}$ be an enumeration of open rational intervals (open intervals with rational endpoints) contained in some (or other) representative A of a. Then, $Ia = \overline{\bigcup_n (p_n, q_n)}$*

Proof. The proof is similar to the proof of Proposition 3.3.11. We need to show that $\overline{\bigcup_n (p_n, q_n)} = sup\{c \in \mathsf{G} \mid c \leq a\}$. Suppose that $c \in \mathsf{G}$ and $c \leq a$. Then $c = \overline{C}$ for some open representative C and $C \subseteq A$ for some representative A of a (see Lemma 3.3.2). Since C is open, C can be written as the union of open rational intervals contained in C. Each such interval is also contained in A, so $C \subseteq \bigcup_n (p_n, q_n)$, and $c \leq \overline{\bigcup_n (p_n, q_n)}$. This shows $\overline{\bigcup_n (p_n, q_n)}$ is an upper bound on $\{c \in \mathsf{G} \mid c \leq a\}$. Now suppose that $b = \overline{B}$ is an upper bound on $\{c \in \mathsf{G} \mid c \leq a\}$. Then, for each $n \in \mathsf{N}$, $\overline{(p_n, q_n)} \leq b$, and $(p_n, q_n) \subseteq B \cup N_n$ for some $N_n \in Null$. So $\bigcup_n (p_n, q_n) \subseteq B \cup \bigcup_n N_n$ and $\overline{\bigcup_n (p_n, q_n)} \leq b$. This shows that $\overline{\bigcup_n (p_n, q_n)}$ is the least upper bound on $\{c \in \mathsf{G} \mid c \leq a\}$. □

We state without proof an obvious corollary which represents the interior in M in terms of open sets rather than rational intervals:

Corollary 3.3.16. *For any $a \in \mathsf{M}$,*

$$Ia = \overline{\bigcup \{O \text{ open} \mid O \subseteq A \text{ for some representative } A \text{ of } a\}}$$

Note from Corollary 3.3.16 that $Ia \in \mathsf{G}$ for any $a \in \mathsf{M}$. Thus, as expected, boxed formulas (*i.e.*, formulas of the form $\Box\varphi$ for some $\varphi \in L$) are evaluated to open elements in M.

3.4 Invariance maps

Our aim in what follows will be to transfer completeness of $S4$ from fnite topologies (= fnite $S4$ Kripke frames) to the measure algebra M by means of truth-preserving maps. In this section, then, we study truth-preserving maps between topological Boolean algebras. In the special case where we deal with topological felds of sets, the key notion is that of an *interior, surjective* map. The key notion in the more general algebraic semantics is that of an *embedding*. The relevant defnitions are given below.

Defnition 3.4.1. *Let A_1 and A_2 be TBA's. A function $\pi : \mathsf{A}_1 \to \mathsf{A}_2$ is a* homomorphism *if it preserves Boolean operations and the interior operator:*

$$\pi(a \vee b) = \pi(a) \vee \pi(b)$$
$$\pi(a \wedge b) = \pi(a) \wedge \pi(b)$$
$$\pi(-a) = -\pi(a) \quad \pi(Ia)$$
$$= I(\pi(a))$$

[11] *We say that* $\pi : \mathsf{A}_1 \to \mathsf{A}_2$ *is an* embedding *if* π *is an injective homomorphism. Finally, we say that* $\pi : \mathsf{A}_1 \to \mathsf{A}_2$ *is an* isomorphism *if* π *is a surjective embedding.*

Lemma 3.4.2. *Suppose that* A_1 *and* A_2 *are TBA's and that* $\pi : \mathsf{A}_1 \to \mathsf{A}_2$ *is a homomorphism. Let* $V^0 : \mathsf{P} \to \mathsf{A}_1$ *be any valuation over* A_1 *and defne the valuation* $V : \mathsf{P} \to \mathsf{A}_2$ *by* $V(P) = \pi \circ V^0(P)$*. Then for any formula* α *in the propositional modal language, L,*

$$V(\alpha) = \pi \circ V^0(\alpha)$$

If π *is an embedding, then (also)*

$$V^0(\alpha) = 1_{\mathsf{A}_1} \textit{ iff } V(\alpha) = 1_{\mathsf{A}_2}$$

Proof. The proof is by induction on the complexity of α. The base case is true by defnition of V, and we prove only the modal clause:

$$\begin{aligned} V(\mathrm{D}\varphi) &= I(V(\varphi)) \\ &= I(\pi \circ V^0(\varphi)) \quad \text{(by inductive hypothesis)} \\ &= \pi(I(V^0(\varphi))) \quad \text{(since } \pi \text{ a homomorphism)} \\ &= \pi \circ V^0(\mathrm{D}\varphi) \end{aligned}$$

For the second part of the lemma (where π is an embedding), note that if $V(\alpha) = 1_{\mathsf{A}_2}$, then by the previous part, $\pi \circ V^0(\alpha) = 1_{\mathsf{A}_2}$. But since π is injective, $V^0(\alpha) = 1_{\mathsf{A}_1}$. Conversely, if $V^0(\alpha) = 1_{\mathsf{A}_1}$, then $V(\alpha) = \pi \circ V^0(\alpha) = \pi \circ 1_{\mathsf{A}_1} = 1_{\mathsf{A}_2}$. □

Let X and Y be topological spaces. Recall that a map, $f : X \to Y$ is *continuous* if the inverse image of every open set in Y is open in X. f is *open* if the image of any open set in X is open in Y. A map that is both open and continuous is called *interior*.

Lemma 3.4.3. *Let* X *and* Y *be topological spaces, and form the corresponding topological feld of sets* $B(X)$ *and* $B(Y)$*. If* $g : X \to Y$ *is interior and surjective, then* $[g^{-1}] : B(Y) \to B(X)$[12] *is an embedding.*

[11] In the fnal equation, 'I' on the LHS is the interior operator in A_1 and 'I' on the RHS is the interior operator in A_2. We trust that the slight abuse of notation here will not confuse.

[12] The map $[g^{-1}]$ is defned on $B(Y)$. It takes subsets of Y to their pullbacks in X—*i.e.*, for $S \subseteq Y$, $[g^{-1}](S) = \{x \in X \mid g(x) \in S\}$.

Proof. Suppose $S_1, S_2 \in B(Y)$, with $S_1 \neq S_2$. WLOG, let $y \in S_1$, $y \notin S_2$. Then since g is surjective, there exists $x \in X$ with $g(x) = y$. But then $x \in [g^{-1}](S_1)$ and $x \notin [g^{-1}](S_2)$, proving that $[g^{-1}]$ is injective.

We need to show that $[g^{-1}]$ preserves the algebraic operations. The Boolean operations are straightforward and we prove only the modal clause: *i.e.*, for any $a \in B(Y)$,

$$[g^{-1}](Ia) = I([g^{-1}](a))$$

By continuity of g, we know that $[g^{-1}](Ia)$ is open in X. Moreover, since $Ia \subseteq a$, we have $[g^{-1}](Ia) \subseteq [g^{-1}](a)$. Thus $[g^{-1}](Ia)$ is an open subset of $[g^{-1}](a)$. To see that it is the largest such subset, suppose $O \subseteq [g^{-1}](a)$ is open in X. Then, since g is open, $g(O)$ is an open subset of a, hence $g(O) \subseteq Ia$. But then $O \subseteq [g^{-1}](Ia)$. □

Proposition 3.4.4. *Suppose that X and Y are topological spaces and $g : X \to Y$ is an interior, surjective map. Let $V' : P \to B(Y)$ be a valuation function and define $V = [g^{-1}] \circ V'$. Then for every formula α of L we have:*

$$V(\alpha) = [g^{-1}] \circ V'(\alpha)$$

and

$$V'(\alpha) = 1_{B(Y)} \text{ iff } V(\alpha) = 1_{B(X)}$$

Proof. Immediate from the previous two lemmas. □

We want to construct embeddings not just from one topological field of sets into another, but from a topological field of sets into the Lebesgue measure algebra M. Such maps will allow us to transfer completeness from a given topological space, or class of spaces, to M. To this end, let us define a new, measure-theoretic property of maps between topological spaces.

Definition 3.4.5. *Let X be the real interval, [0,1], let μ be standard Lebesgue measure on X, and let Y be a topological space. We say that a function $g : X \to Y$ has the M-property if for every subset $S \subseteq Y$,*

(i) *$g^{-1}(S)$ is Lebesgue-measurable.*

(ii) *For any open set $O \subseteq X$, if $g^{-1}(S) \cap O \neq \emptyset$, then $\mu(g^{-1}(S) \cap O) > 0$.*

Proposition 3.4.6. *Let X, μ, and Y be as in Definition 3.4.5. Suppose $g : X \to Y$ is an interior, surjective map, and that g satisfies the M-property. Then the function $\Phi : B(Y) \to M$ defined by:*

$$\Phi(S) = \overline{g^{-1}(S)}$$

(for any $S \subseteq Y$) is an embedding.[13]

Proof. We need to show that Φ preserves Boolean operations, the interior operator, and is injective. The Boolean cases are straightforward and we leave them to the reader. For the interior operator, we need to show that

$$\Phi(I(S)) = I(\Phi(S))$$

We know that:

$$\begin{aligned}
\Phi(I(S)) &= \overline{g^{-1}(Int(S))} \\
&= \overline{Int(g^{-1}(S))} \qquad \text{(since } [g^{-1}] \text{ is a homomorphism)} \\
&= \overline{\bigcup \{O \text{ open} \mid O \subseteq g^{-1}(S)\}} \quad \text{(by defnition of interior)} \\
I(\Phi(S)) &= sup\, \{c \text{ open} \mid c \leq \overline{g^{-1}(S)}\} \\
&= \overline{\bigcup \{O \text{ open} \mid O \subseteq g^{-1}(S) \cup N \text{ for some } N \in Null_\mu\}}
\end{aligned}$$

(where the last equality follows from Corollary 3.3.16). So it is suffcient to show that for any open set $O \subseteq X$, if $O \subseteq g^{-1}(S) \cup N$ for some $N \in Null_\mu$, then $O \subseteq g^{-1}(S)$.

Suppose not. Then there exists $O \subseteq X$ open such that $O \subseteq g^{-1}(S) \cup N$ for some $N \in Null_\mu$ but $O \not\subseteq g^{-1}(S)$. So there exists $x \in O$ such that $x \notin g^{-1}(S)$. Thus $x \in N$. Now let $g(x) = y \in Y$. We know that $O \cap g^{-1}(y) \neq \emptyset$. And since g has the M-property, $\mu(O \cap g^{-1}(y)) > 0$. But $y \notin S$, and $O \subseteq g^{-1}(S) \cup N$. It follows that $O \cap g^{-1}(y) \subseteq N$. This contradicts the fact that N has measure zero.

$\perp$.

It remains to show only that Φ is injective. Suppose that $\Phi(S_1) = \Phi(S_2)$. Then $g^{-1}(S_1) = g^{-1}(S_2)$. So $\mu(g^{-1}(S_1) \triangle g^{-1}(S_2)) = 0$. But $g^{-1}(S_1) \triangle g^{-1}(S_2) = g^{-1}(S_1 \triangle S_2)$. So $\mu(g^{-1}(S_1 \triangle S_2)) = 0$. Now it follows from the fact that g has the M-property that for any non-empty set $S \subseteq Y$, we have $\mu(g^{-1}(S)) > 0$. (Take as the open set O in Defnition 3.4.5 (*ii*) the entire space, [0,1].) This means that $S_1 \triangle S_2 = \emptyset$. So $S_1 = S_2$. $\square$

Corollary 3.4.7. *Let X, μ, Y, g, and Φ be as above. Suppose that $V^0 : P \to B(Y)$ is a valuation, and defne the valuation, $\overline{V} = \Phi \circ V^0$. Then for every formula a in L we have:*

$$\overline{V}(a) = \Phi \circ V^0(a)$$

[13] Note that $\Phi = q \circ g^{-1}$, where q is the restriction of the quotient map from $B([0,1])$ to M_Φ to the set $\{g^{-1}(S) \mid S \subseteq Y\}$. We know that g^{-1} is an embedding, but the (unrestricted) quotient map is *not* an embedding. What the proposition shows is that when we restrict the quotient map to the collection of g-pullbacks of subsets of Y, then the resulting map *is* an embedding.

and

$$\overline{V}(a) = 1_{\mathbf{M}} \text{ iff } V^0(a) = 1_{B(Y)}$$

Proof. Immediate from Lemma 3.4.2 and Proposition 3.4.6. □

Remark 3.4.8. *Let B be a subset of the real interval [0, 1] of measure 1 with the relative topology, and let Y, μ, g, and Φ be as above, except that g is now defned on B. Then we can still view Φ as an embedding of the algebra $B(Y)$ in $\mathbf{M}$ – even though, strictly speaking, these defnitions are not correct (*i.e.*, well defned). That is because the measure algebra $Leb([B]) \setminus Null_\mu$ is isomorphic to the Lebesgue measure algebra, $\mathbf{M}$. Thus it is suffcient, in Corollary 3.4.7, to require that g be defned only on a subset of [0,1] of measure equal to 1. This will make life simpler for us in the next section, where we aim to construct such a map, g.*

3.5 Completeness of $S4$ for the Lebesgue measure algebra

We know that the logic $S4$ is complete for the class of fnite $S4$ Kripke frames (= fnite topologies).[14] Algebraically put, $S4$ is complete for the class of topological Boolean algebras, $\{B(\mathbf{F}) \mid \mathbf{F}$ is a fnite topology$\}$. Our aim in this section is to leverage this nice result toward a proof of completeness of $S4$ for $\mathbf{M}$. Our strategy will be to embed such Kripke frames in the algebra $\mathbf{M}$. To do this, we need to construct 'nice' maps from the real interval [0, 1] (or, more precisely, a subset, B, of the interval of measure 1) to the Kripke frame $\mathbf{F}$. In particular, we need to construct a map, $g : B \to \mathbf{F}$ that satisfes the conditions of Proposition 3.4.6. We begin by recalling the *thick Cantor sets*, which will play a crucial role in the construction of our map, g.

3.5.1 Thick Cantor sets

Recall the construction of the (normal) Cantor set. We begin with the interval $[0, 1]$. At stage $n = 0$ of construction, we remove the open middle third $(\frac{1}{3}, \frac{2}{3})$, leaving "remaining intervals" $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. At stage $n = 1$, we remove the open middle thirds of each of these intervals, $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$, leaving remaining intervals $[0, \frac{1}{9}]$, $[\frac{2}{9}, \frac{1}{3}]$, $[\frac{2}{3}, \frac{7}{9}]$ and $[\frac{8}{9}, 1]$, and so on. In general, at stage $n + 1$ of construction, we remove the open middle thirds of each remaining interval from stage n. The Cantor set, C, is the set of points remaining after infnitely many

[14] In the remainder of the chapter, I will use 'fnite $S4$ Kripke frames' and 'fnite topologies' interchangeably.

Figure 10: First fve stages of construction of the Smith-Volterra-Cantor set, *K*.

stages of construction. To calculate the measure of *C*, we need only subtract the total measure of intervals removed from the measure of the unit interval, [0,1]:

$$1 - \sum_{n\geq 0} 2^n \left(\frac{1}{3}\right)^{n+1} = 1 - \frac{1}{3} \sum_{n\geq 0} \left(\frac{2}{3}\right)^n = 0$$

An easy argument due to Dana Scott shows that removing middle *fourths*, *ffths*, *etc.* (as opposed to middle *thirds*) does not affect the measure of *C*. Indeed, let C_n be the set resulting from removing open middle intervals of proportional length $1/n$ at each stage of construction. After removing the frst middle interval we produce scaled copies of C_n on the intervals $[0, \frac{n-1}{2n}]$ and $[\frac{n+1}{2n}, 1]$, giving,

$$m(C_n) = 2\,\frac{n-1}{2n}\, m(C_n)$$

and $m(C_n) = 0$.

We can, however, construct a set that is 'Cantor-like' with non-zero measure. The trick is to remove successively smaller portions of remaining intervals. The set we end up with is sometimes called a 'thick' or 'fat' Cantor set. The particular version of it below has measure $= 1/2$, but this is not necessary — sets of arbitrary positive measure can be constructed in similar fashion.[15]

Defnition 3.5.1. *Begin with the interval* $[0, 1]$*, and at stage* $n = 0$ *of construction, remove the open middle interval of length* $\frac{1}{4}$ *leaving remaining intervals* $[0, \frac{3}{8}] \cup [\frac{5}{8}, 1]$. *At stage* $n = 1$*, remove open middle* $\frac{1}{16}$*'s from each interval, leaving* $[0, \frac{5}{32}] \cup [\frac{7}{32}, \frac{3}{8}] \cup [\frac{5}{8}, \frac{25}{32}] \cup [\frac{27}{32}, 1]$, etc. *In general, at stage* n *of construction, remove open middle intervals of length* $(\frac{1}{4})^{n+1}$ *from each remaining interval. The set of points remaining after infnitely many stages of construction is the Smith-Volterra-Cantor set. We call it the 'thick' Cantor set and denote it by* K.[16] *(See Figures 10 and 11.)*

[15] To construct a thick Cantor set with measure $1-c$, remove middle intervals of length $2c(\frac{1}{4})^{n+1}$ at stage n of construction. Over the course of the construction we remove a total measure of $2c\sum_{n\geq 0} 2^n(\frac{1}{4})^{n+1} = 2c\sum_{n\geq 0}(\frac{1}{2})^{n+2} = 2c(\frac{1}{2}) = c$.

[1] Figures 10 and 11 are licensed by Creative Commons.

Figure 11: The set K. After white intervals have been removed, the black points which remain make up K.

What is the measure of K? Note that at each fnite stage n of construction of K, 2^n intervals of length $(\frac{1}{4})^{n+1}$ are removed, so the total measure of points removed is

$$\sum_{n\geq 0} 2^n (\frac{1}{4})^{n+1} = \sum_{n\geq 0} (\frac{1}{2})^{n+2} = \frac{1}{2}$$

and $m(K) = 1 - 1/2 = 1/2$.

Proposition 3.5.2. *Let O be an open set with $K \cap O \neq \emptyset$. Then $K \cap O$ has non-zero measure.*

Proof. Let O be open and $x \in K \cap O$. Then, since $x \in K$, x is in a remaining interval at each stage of construction of K. Let $R_{n,x}$ denote the remaining interval containing x at stage n of construction. The length of remaining intervals tends to zero, so for N large enough, $R_{N,x} \subseteq O$. But, by symmetry, $m(K \cap R_{N,x}) = (\frac{1}{2})^{N+2} > 0$. (At stage N of construction, there are 2^{N+1} remaining intervals and they split the measure of K equally). Thus

$$m(K \cap O) \geq m(K \cap R_{N,x}) > 0$$

□

We can construct a 'scaled copy' of K by starting from the interval $[a, b]$ instead of $[0, 1]$, and successively removing middle segments of length $(b - a)(\frac{1}{2})^{2n+2}$. In fact, we can carry out the construction of K on any closed, open, or half-open interval $[a, b], (a, b), [a, b), (a, b]$. If we start from the open interval (a, b), the resulting set is not closed (compact, *etc.*) and hence differs in important topological properties from K. Nevertheless, with slight abuse of notation, we refer to all such constructions as 'scaled copies' of K. Clearly the measure of a scaled copy of K on any of the intervals $[a, b], (a, b), [a, b), (a, b]$ is just $\frac{1}{2}(b - a)$.

We state without proof an obvious corollary to Proposition 3.5.2:

Corollary 3.5.3. *Let K^* be a scaled copy of K. If O is open and $O \cap K^*$ is non-empty, then $O \cap K^*$ has non-zero measure.*

We are now, fnally, in a position to prove Proposition 3.3.10, which states that $\mathbb{M} \neq \mathbb{G}$ (see 3.3). The example is due to Dana Scott, but we give a different proof here.

Proof of Proposition 3.3.10. We claim that $\overline{K} \notin \mathbb{G}$ (and thus $\mathbb{M} \neq \mathbb{G}$). We need to show that for any open set O, $K \not\sim O$. Suppose $O \subseteq [0, 1]$ is open and $O \sim K$. We know $O \cap K \neq \emptyset$ (else $K \subseteq O \triangle K$ and $O \not\sim K$). Let $x \in O \cap K$. By the proof of Proposition 3.5.2, there exists $N \in \mathbb{N}$ with $R_{N,x} \subseteq O$ (where $R_{n,x}$ is, again, the remaining interval at stage n containing x). But at stage $n + 1$ of construction of K, we remove from $R_{N,x}$ an open interval, I, of non-zero measure. So $I \subseteq O - K$ and and $K \not\sim O$. $\perp$. □

3.5.2 Construction of a truth preserving map

We now construct the map g mentioned above, that will transfer completeness from fnite topological spaces (= fnite Kripke $S4$ frames) to the Lebesgue measure algebra.

Let $\mathcal{F} = \langle U, R, w_0 \rangle$ be a fnite rooted $S4$ Kripke frame (= fnite topology), where $U = \{w_0, \ldots, w_m\}$.

Preliminary to constructing the map, g, we defne a sequence of approximating functions, g_i $(i \in \mathbb{N})$.

We begin by constructing g_0. Recall the construction of the thick Cantor set, K, given above. We will denote the union of open intervals removed at the nth stage of construction of K by O_n $(n \geq 0)$. Now we put,

$$g_0(x) = \begin{cases} w_s & \text{if } x \in O_n \text{ and } n = s \pmod{m} \\ w_0 & \text{otherwise} \end{cases}$$

Note that g_0 labels each point in the thick Cantor set, K, by w_0, and that all other points belong to some open interval that is uniformly labeled under g_0 by some node (or other) in U. If I is a maximal such interval (*i.e.*, there does not exist an open interval I' such that $I \subset I'$ and I' is uniformly labeled under g_0), then we call I a 'removed interval under g_0.' This completes our construction of g_0.

Now suppose that the function g_i is defned on every point in [0,1], and that under g_i there is some countable collection of disjoint open intervals uniformly labeled under g_i by some node (or other) in U. Moreover, assume that each of these intervals, I, is maximal in the sense specifed above (*i.e.*, there does not exist an open interval I' such that $I \subset I'$ and I' is uniformly labeled under g_i). We call these intervals the 'removed intervals of g_i.' For each such interval, I, uniformly labeled by $w_k \in U$, we now put $U_k = \{w \in U \mid w_k R w\}$ and we denote by n_k

the cardinality of U_k. Finally, we order the elements of U_k in some way, putting $U_k = \{u_1, u_2, \ldots, u_{m_k}\}$. We now repeat the construction given above, on the interval I. That is, denoting by O_n the union of open intervals removed at stage n of the construction of $K(I)$, we let, for $x \in I$:

$$g_{i+1}(x) = \begin{cases} u_s & \text{if } x \in O_n \text{ and } n = s \pmod{m_k} \\ w_k & \text{otherwise} \end{cases} \tag{3.6}$$

For all $x \in [0,1]$ such that x does not belong to a removed interval of g_i, we put $g_{i+1}(x) = g_i(x)$.

Note that under g_{i+1} there is a countable collection of maximal (in the sense defned above) uniformly labeled open intervals. We call these intervals the 're-moved intervals of g_{i+1}.' This completes our construction of the maps, g_i ($i \in \mathbb{N}$).

Note that some points $x \in [0, 1]$ belong to a 'removed interval of g_i' for each $i \in \mathbb{N}$. We denote the collection of all such points by L. We denote the collection of all other points in [0,1] by B. Thus, the interval [0,1] is the disjoint union of L and B.

For each $x \in B$, there exists $i \in \mathbb{N}$ such that

$$\text{for all } j \geq i,\ g_j(x) = g_i(x)$$

Let us denote the least such i by i_x.

We are now ready to defne the function $g : B \to \mathbb{F}$ as follows:

$$g(x) = g_{i_x}(x) \tag{3.7}$$

for all $x \in B$.

3.5.3 Completeness proof

We need to show that the map $g : B \to \mathbb{F}$ defned in the previous section satisfes the conditions of Proposition 3.4.6. In other words, we need to show that g is interior, surjective and satisfes the M-property. Also, we need to show that the measure of the set B is 1. The work of this section is devoted to that end.

In what follows, let g_i ($i \in \mathbb{N}$), g, B and L be as defned in the previous section.

Lemma 3.5.4. *For all $x \in [0, 1]$, $i \in \mathbb{N}$,*

$$g_i(x) R g_{i+1}(x)$$

Proof. By construction of g_i. □

Lemma 3.5.5. *Suppose $x \in B$, $g(x) = w$, and wRw'. Then for any $\varepsilon > 0$, there exists $y \in B$ such that $|x - y| < \varepsilon$ and $g(y) = w'$.*

Proof. Let $x \in B$, $g(x) = w$, and wRw'. Then since $x \in B$, there exists $i \in \mathbb{N}$ such that for all $j \leq i$, x belongs to a removed interval of g_j and x does not belong to any removed interval of g_{i+1}. By construction, this means that $x \in K(I)$ for some removed interval I of g_i. But then x belongs to some remaining interval $R_{n,x}$ at each stage n of construction of $K(I)$, and as we know, $length\ (R_{n,x}) \to 0$. It follows that for N large enough, $R_{N,x} \subseteq B(x, \varepsilon)$, where $B(x, \varepsilon)$ is the open interval centered at x with radius ε. But now, by construction of g_{i+1}, there exists a removed interval I' of g_{i+1}, with $I' \subseteq R_{N,x}$ and $g_{i+1}(I') = w'$. (To see this, consider all the intervals removed during construction of the scaled thick Cantor set $K(I)$ between stages N and $N + m$ of construction.) Again, by construction of g_{i+2}, for any $y \in K(I')$, $g(y) = w'$. Thus we have,

$$y \in I' \subseteq R_{N,x} \subseteq B(x, \varepsilon)$$

and $g(y) = w'$, as desired. □

Lemma 3.5.6. $\mu(L) = 0$. $\mu(B) = 1$.

Proof. Let S_i be the union of removed intervals of g_i ($i \in \mathbb{N}$). Then,

$$S_i \supseteq S_{i+1}$$

and

$$L = \bigcap_{i \in \mathbb{N}} S_i$$

and $m(S_0) = 1$. It follows that $m(L) = lim_{i\to\infty} m(S_i)$. But $m(S_{i+1}) = \frac{1}{2} m(S_i)$.[17] So $m(S_i) \to 0$, and $m(L) = 0$. Now we have $m(B) = m([0, 1]) - m(L) = 1$. □

Proposition 3.5.7. *g is continuous.*

Proof. Let U be an open subset of F, and suppose $x \in g^{-1}(U)$. Then $x \in B$, so by construction, there is some $i \in \mathbb{N}$ such that x belongs to a removed interval of g_j for all $j \leq i$, and x does not belong to a removed interval of g_{i+1}. Let $I_{i,x}$ be the removed interval of g_i containing x. Then by construction of stagewise labeling functions, for each $y \in I_{i,x}$ we have $g_i(y) = g_i(x) \in U$. It follows from Lemma **3.5.4** that for each point $y \in I_{i,x} \cap B$, $g(y) \in U$. So $x \in I_{i,x} \cap B \subseteq g^{-1}(U)$. But $I_{i,x} \cap B$ is open in B. Thus $g^{-1}(U)$ is open. □

[17] By construction, and since $m(K(I)) = \frac{1}{2} m(I)$.

Proposition 3.5.8. *g is open.*

Proof. Let $O \subseteq B$ be open, and let $w \in g(O)$. Then there exists $x \in O$ such that $w = g(x)$. Suppose wRw'. By Lemma 3.5.5, there exists $y \in B$ such that $y \in O$, and $g(y) = w'$. Thus $U_w \subseteq g(O)$, and $g(O)$ is open. □

Proposition 3.5.9. *g is surjective.*

Proof. This follows from the fact that g is open and g 'hits' the root, w_0, of F (*i.e.*, there exists $x \in B$ with $g(x) = w_0$.) □

Proposition 3.5.10. *g has the M-property.*

Proof. (i) To see that for any set $S \subseteq U$, $g^{-1}(S)$ is Lebesgue measurable, let $w \in U$. Note that by construction of g, $g^{-1}(w)$ is a countable union of scaled copies of thick Cantor sets, $K(I)$. Thus $g^{-1}(w)$ is a countable union of Borel sets, hence Borel. Since S is finite, $g^{-1}(S)$ is a finite union of Borel sets, hence Borel. (ii) We need to show that for any open set $O \subseteq [0, 1]$ and $S \subseteq U$, if $g^{-1}(S) \cap O \neq \emptyset$, then $\mu(g^{-1}(S) \cap O) > 0$. It is sufficient to prove this for the case where $S = \{w\}$ for some $w \in U$. Thus suppose $O \subseteq [0, 1]$ is open, and for some $w \in U$, $g^{-1}(w) \cap O \neq \emptyset$. Then there exists $x \in g^{-1}(w) \cap O$. Since $x \in B$, there exists $i \in I$ such that x belongs to a removed interval of g_j for all $j \leq i$, and x does not belong to a removed interval of g_{i+1}. By construction of the stagewise labeling functions, $x \in K(I)$ for some removed interval I of g_i[18], and for every $y \in K(I)$, $g(y) = g(x) = w$. So $K(I) \subseteq g^{-1}(w)$. But since $x \in K(I)$, we know $O \cap K(I) \neq \emptyset$. By Corollary 3.5.3, $\mu(O \cap K(I)) > 0$. Now we have $O \cap K(I) \subseteq O \cap g^{-1}(w)$. So

$$\mu(O \cap g^{-1}(w)) \geq \mu(O \cap K(I)) > 0$$

□

We now define the function $\Phi : \mathcal{B}(\mathsf{F}) \to \mathsf{M}$ by

$$\Phi(S) = \overline{g^{-1}(S)}$$

for all $S \subseteq \mathsf{F}$.

Proposition 3.5.11. *Φ is an embedding.*

Proof. Immediate from Proposition 3.4.6, Proposition 3.5.7, Proposition 3.5.8, Proposition 3.5.9, and Proposition 3.5.10. □

[18] where $K(I)$ is, again, the scaled copy of the thick Cantor set, K, on the interval, I

Proposition 3.5.12. *Suppose that* $V' : P \to B(F)$ *is a valuation, and defne the valuation,* $V = \Phi \circ V'$*, over the algebra* M*. Then for every formula* a *in* L *we have:*

$$\overline{V}(a) = \Phi \circ \overline{V'}(a)$$

and

$$\overline{V}(a) = 1_{\mathsf{M}} \text{ iff } \overline{V'}(a) = 1_{B(F)}$$

Proof. Immediate from Lemma 3.4.2, and Proposition 3.5.11. □

Theorem 3.5.13. $S4$ *is complete for the Lebesgue measure algebra,* M.

Proof. Let a be a non-theorem of $S4$ (*i.e.*, $\nvdash_{S4} a$). Then a is refuted in some fnite Kripke frame, F. That is, there is some algebraic model, $\langle B(F), V' \rangle$ such that $\overline{V'}(a) \neq 1_{B(F)}$. We defne the algebraic model $\langle \mathsf{M}, V \rangle$, letting $V = \Phi \circ V'$, where Φ is as defned above. By Proposition 3.5.12, $\overline{V}(a) \neq 1_{\mathsf{M}}$, and a is refuted in M. We have shown that for any a in the language L,

$$\nvdash_{S4} a \;\Rightarrow\; \not\models_{\mathsf{M}} a$$

□

We close this chapter by proving two interesting corollaries of the above theorem.

We know, from Tarski's proof of completeness of $S4$ for the reals, that any non-theorem, a, of $S4$ can be refuted at a point in the real interval, *i.e.*, there is a valuation, $V : P \to B([0, 1])$, and point $x \in [0, 1]$ with $x \notin \overline{V}(a)$. The next corollary states that if a is a non-theorem of $S4$, there exists a valuation, $V : P \to B([0, 1])$, that refutes a at each point in a subset of $[0, 1]$ of measure *arbitrarily close to 1.*

Corollary 3.5.14. *Suppose* a *is a non-theorem of* $S4$*. Then for any* $E > 0$*, there exists a valuation,* $V : P \to B([0, 1])$*, with* $\mu(\overline{V}(a)) < E$*. Likewise, for any* $E > 0$ *there exists a valuation* $V^* : P \to \mathsf{M}$*, and an element* $s \in \mathsf{M}$*, with* $m(s) < E$ *and* $\overline{V_*}(a) = s$

Proof Sketch. Let a be a non-theorem of $S4$, and let $E > 0$. Then a is refuted in some model, $M = \langle B(F), V' \rangle$, where F is a fnite Kripke frame. We defne an embedding, $\Phi^* : B(F) \to \mathsf{M}$, using thick Cantor sets of measure $1 - E$, but otherwise identical to Φ. Let K^* be the thick Cantor set of measure $1 - E$. Then stagewise labeling functions, g_i^*, are constructed as before (see (3.6) above) but using K^* instead of K. Again, let g^* be the limit of stagewise labeling functions, g_i^* (see (3.7) above). Now g^* is an interior, surjective map. We defne the valuation

$V : \mathsf{P} \to B([0,1])$ by putting $V(P) = g^{*-1} \circ V^0$, and by Proposition 3.4.4 we have, for any formula φ in the language L, $V(\varphi) = g^{-1} \circ V^0(\varphi)$. The reader can now verify that $K^* \subseteq g^{-1}(V^0(\neg a)) = V(\neg a)$. It follows that $\mu(V(\neg a)) \geq 1 - E$, and $\mu(V(a)) < E$

For the second part of the corollary, we define the valuation $\overline{V} : \mathsf{P} \to \mathsf{M}$ by $V(P) = \Phi^* \circ V^0$. Again, Φ^* is an embedding[19], and so for any formula φ in the language L we have $V(\varphi) = \Phi^* \circ V^0$. The reader can again verify that $K^* \leq \Phi^* \circ V^0(\neg a) = V(\neg a)$. It follows that $\mu(V(\neg a)) \geq 1 - E$, and $\mu(V(a)) < E$ 2 □

As a final corollary, we prove that Intuitionistic propositional logic (IPC) is complete for the frame G. Let the propositional language L_0 consist of a countable set, $\mathsf{P} = \{P_n \mid n \in \mathsf{N}\}$ of atomic variables and be closed under binary connectives $\to, \vee, \wedge$ and unary operator $\neg$. Recall that G is a complete Heyting algebra. In particular, for any elements $x, y \in \mathsf{G}$, there exists an element, $x \Rightarrow y \in \mathsf{G}$, called the *relative pseudo-complement* of x with respect to y and defined by:

$$sup\{c \in \mathsf{G} \mid c \wedge x \leq y\}$$

Let $V : \mathsf{P} \to \mathsf{G}$ be a valuation assigning propositional variables to arbitrary elements of G. We extend V by the following recursive clause:

$$V(\varphi \to \psi) = V(\varphi) \Rightarrow V(\psi)$$

(For the remaining connectives: V is defined in the usual way on $\{\&, \vee\}$, '$\neg\varphi$' abbreviates '$\varphi \to \perp$' and '$\varphi \leftrightarrow \psi$' abbreviates '$\varphi \to \psi \,\&\, \psi \to \varphi$'.)

For any formula $\Phi \in L_0$, let $T(\varphi)$ be the Gödel-Tarski translation of φ given inductively as follows:

$T(P) = \Box P$ for all propositional variables P

$T(\perp) = \perp$

$T(\varphi \vee \psi) = T(\varphi) \vee T(\psi)$

$T(\varphi \wedge \psi) = T(\varphi) \wedge T(\psi)$

$T(\varphi \to \psi) = \Box(T(\varphi) \to T(\psi))$

Gödel and Tarski showed that $\vdash_{IPC} a$ iff $\vdash_{S4} T(a)$ for any formula $a \in L_0$. Moreover, for any valuation $V : \mathsf{P} \to \mathsf{M}$, we can define the valuation, $V_I : \mathsf{P} \to \mathsf{G}$,

[19] One has to check, here, that when we use Cantor sets of measure $1 - c$, Lemma 3.5.6 still holds. We do not carry out the relevant calculation here, but leave it to the reader.

[20] Here μ is used to denote both the standard Lebesgue measure on the reals, and the measure on the Lebesgue measure algebra, M. We trust that this does not lead to undue confusion.

by $V_I(P) = V(\mathrm{D}P)$. It is easy to show that for any formula, $\alpha \in L_0$, $T(\alpha)$ L_1 and

$$V_I(\alpha) = V(T(\alpha))$$

[21] In particular, $V(T(\alpha)) \in \mathsf{G}$ for each α L_0 (the Gödel translation of any formula is evaluated to an open element).

Corollary 3.5.15. *IPC is complete for* G.

Proof. Suppose $0_{IPC}\ \alpha$. Then $0_{S4}\ T(\alpha)$. By completeness of $S4$ for M, there is a valuation $V : \mathsf{P} \to \mathsf{M}$ with $V(T(\alpha))\ 6= [0, 1]$. But letting V_I be defned as above, we have $V_I(\alpha) = V(T(\alpha)) = [0, 1]$, so α is refuted under V_I in G. □

[21]The proof is by induction on the complexity of α.

Chapter 4

Probabilistic Semantics for Dynamic Topological Logic

Abstract. In this chapter we extend Dana Scott's probabilistic semantics for the basic propositional modal language to a more complex modal language with two independent modalities. In particular, we give a probabilistic semantics for basic *dynamic topological logic*. Dynamic topological logics were introduced in the 1990's as a way of describing dynamic space, or a topological space together with a continuous function acting on the space. The simplest dynamic topological logic is *S*4*C*, which has both the usual necessity modality, 'D', and a new temporal modality, 'Q́ We extend Scott's probabilistic semantics to this bimodal logic. The main result of the chapter is that *S*4*C* is complete for the Lebesgue measure algebra. A strengthening of this result, also proved here, is that there is a *single* probabilistic model in which all non-theorems of S4C are refuted.

4.1 Introduction

Kripke frames for normal modal logics, consisting of a set of possible worlds together with a binary accessibility relation, are, by now, widely familiar. But long before Kripke semantics became standard, Tarski showed that the propositional modal logic $S4$ can be interpreted in topological spaces. In the topological semantics for $S4$, a topological space is fxed, and each propositional variable, P, is assigned to an arbitrary subset of the space: the set of points where P is true. Conjunctions, disjunctions and negations are interpreted as set-theoretic intersections, unions and complements (thus, *e.g.*, '$\varphi \wedge \psi$' is true at all points in the intersection of the set of points where 'φ' is true and the set of points where 'ψ' is true.) The 'D'-modality of $S4$ is interpreted via the topological interior: 'Dφ' is true at any point in the topological interior of the set of points at which 'φ' is true.

In this semantics, the logic $S4$ can be seen as describing topological spaces. Indeed, with the topological semantics it became possible to ask not just whether $S4$ is complete for the set of topological validities—formulas valid in *every* topological space—but also whether $S4$ is complete for any *given* topological space. The culmination of Tarski's work in this area was a very strong completeness result. In 1944, Tarski and McKinsey proved that $S4$ is complete for any dense-in-itself metric space. One particularly important case was the real line, $\mathbb{R}$, and as the topological semantics received renewed interest in recent years, more streamlined proofs of Tarski's result for this special case emerged in, *e.g.*, (5), (18), (26), (29), and (38).

The real line, however, can be investigated not just from a topological point of view, but from a measure-theoretic point of view. Here, the probability measure we have in mind is the usual Lebesgue measure on the reals. In the last several years Dana Scott introduced a new *probabilistic* or *measure-based* semantics for $S4$ that is built around Lebesgue measure on the reals and is in some ways closely related to Tarski's older topological semantics.

Scott's semantics is essentially algebraic: formulas are interpreted in the *Lebesgue measure algebra*, or the σ-algebra of Borel subsets of the real interval [0,1], modulo sets of measure zero (henceforth, "null sets"). We denote this algebra by $\mathbf{M}$. Thus elements of $\mathbf{M}$ are equivalence classes of Borel sets. In Scott's semantics, each propositional variable is assigned to some element $\mathbf{M}$. We say the *value* of the propositional variable P is that element of the algebra to which P is assigned. Conjunctions, disjunctions and negations are interpreted as meets, joins and complements in the algebra, respectively. In order to interpret the $S4$ 'D'-modality, we add to the algebra an *interior* operator (defned below), which we construct from the collection of *open* elements in the algebra, or elements that have an open representative. Unlike the Kripke or topological semantics, there is no notion here of truth *at a point* (or at a “world”). In (11) and in (22) it was shown that $S4$ is complete for the Lebesgue measure algebra.[1]

The introduction of a measure-based semantics for $S4$ raises a host of questions that are, at this point, entirely unexplored. Among them: What about natural extensions of $S4$? Can we give a measure-based semantics not just for $S4$ but for some of its extensions that have well-known topological interpretations?

This chapter focuses on a family of logics called *dynamic topological logics*. These logics were investigated over the last ffteen years, in an attempt to describe “dynamic topological systems” by means of modal logic. A dynamic topological system is a pair $\langle X, f\rangle$ where X is a topological space and f is a continuous function on X. We can think of f as moving points in X in discrete units of time. Thus in the frst moment in time, x is mapped to $f(x)$, in the second moment to $f(f(x))$, and so on. The simplest dynamic topological logic is $S4C$. In addition to the $S4$ ‘D’-modality, it has a temporal modality, which we denote by ‘Q’. Intuitively, we understand the formula ‘$\mathsf{Q}P$’ as saying that “at the next moment in time, P will be true.” Thus we put: $x \in V(\mathsf{Q}P)$ iff $f(x) \in V(P)$. In (19) and (37) it was shown that $S4C$ is incomplete for the real line, R. However, in (38) it was shown that $S4C$ is complete for Euclidean spaces of arbitrarily large fnite dimension, and in (10) it was shown that $S4C$ is complete for R^2.

The aim of this chapter is to give a measure-based semantics for the logic $S4C$, along the lines of Scott’s semantics for $S4$. Again, formulas will be assigned to some element of the Lebesgue measure algebra, M. But what about the *dynamical* aspect—*i.e.*, the interpretation of the ‘Q’-modality? We show that there is a very natural way of interpreting the ‘Q’-modality via operators on the algebra M that take the place of continuous functions in the topological semantics. These operators can be viewed as transforming the algebra in discrete units of time. Thus one element is sent to another in the frst instance, then to another in the second instance, and so on. The operators we use to interpret $S4C$ are O-operators: ones that take “open” elements in the algebra to open elements (defned below). But there are obvious extensions of this idea: for example, to interpret the logic of *homeomorphisms* on topological spaces, one need only look at *automorphisms* of the algebra M.

Adopting a measure-based semantics for $S4C$ brings with it certain advantages. Not only do we reap the probabilistic features that come with Scott’s semantics for $S4$, but the curious *dimensional asymmetry* that appears in the topological semantics (where $S4C$ is *in*complete for R but complete for R^2) disappears in the measure-based semantics. The main result of the chapter is that the logic $S4C$ is complete for the Lebesgue-measure algebra. A strengthening of this result, also

[1]The proofs were arrived at independently and at roughly the same time.

proved here, is that $S4C$ is complete for a *single* model of the Lebesgue measure algebra. Due to well-known results by Oxtoby, this algebra is isomorphic to the algebra generated by Euclidean space of arbitrary dimension. (Indeed, as we show below, it is isomorphic to the reduced measure algebra generated by any separable metric space together with a σ-fnite, non-atomic Borel measure on the space.) In other words, $S4C$ is complete for the reduced measure algebra generated by *any* Euclidean space.

4.2 Topological semantics for $S4C$

Let the language $L_{\mathrm{D},\mathsf{Q}}$ consist of a countable set, $\mathsf{P} = \{p_n \mid n \in \mathsf{N}\}$, of propositional variables, and be closed under the binary connectives $\wedge, \vee, \rightarrow, \leftrightarrow$, unary operators, $\neg, \mathrm{D}, <$, and a unary modal operator Q (thus, $L_{\mathrm{D},\mathsf{Q}}$ is the language of propositional $S4$ enriched with a new modality, Q).

Defnition 4.2.1. *A* dynamic topological space *is a pair* $\langle X, f\rangle$, *where X is a topological space and $f : X \rightarrow X$ is a continuous function on X. A* dynamic topological model *is a triple,* $\langle X, f, V\rangle$, *where X is a topological space, $f : X \rightarrow X$ is a continuous function, and $V : \mathsf{P} \rightarrow \mathsf{P}(X)$ is a valuation assigning to each propositional variable a subset of X. We say that* $\langle X, f, V\rangle$ *is a model* over X.

We extend V to the set of all formulas in $L_{\mathrm{D},\mathsf{Q}}$ by means of the following recursive clauses:

$$V(\varphi \vee \psi) = V(\varphi) \cup V(\psi)$$
$$V(\neg\varphi) = X - V(\varphi)$$
$$V(\mathrm{D}\varphi) = Int\,(V(\varphi))$$
$$V(\mathsf{Q}\varphi) = f^{-1}(V(\varphi))$$

where '*Int*' denotes the topological interior.

Let $N = \langle X, f, V\rangle$ be a dynamic topological model. We say that a formula φ is *satisfed* at a point $x \in X$ if $x \in V(\varphi)$, and we write $N, x \models \varphi$. We say φ is true in N ($N \models \varphi$) if $N, x \models \varphi$ for each $x \in X$. We say φ is *valid* in X ($\models_X \varphi$), if for any model N over X, we have $N \models \varphi$. Finally, we say φ is *topologically valid* if it is valid in every topological space.

Defnition 4.2.2. *The logic $S4C$ in the language $L_{\mathrm{D},\mathsf{Q}}$ is given by the following axioms:*

- *the classical tautologies,*
- $S4$ *axioms for* D.

(A1) $Q(\varphi \vee \psi) \leftrightarrow (Q\varphi \vee Q\psi)$,

(A2) $(Q\neg\varphi) \leftrightarrow (\neg\, Q\, \varphi)$,

(A3) $QD\varphi \rightarrow D\, Q\, \varphi$ *(the axiom of continuity)*

and the rules of modus ponens and necessitation for both D *and* Q. *Following (19), we use* $S4C$ *both for this axiomatization and for the set of all formulas derivable from the axioms by the inference rules.*

We close this section by listing the known completeness results for $S4C$ in the topological semantics.

Theorem 4.2.3. *(Completeness) For any formula* $\varphi \in L_{D,Q}$, *the following are equivalent:*
(i) $S4C \vdash \varphi$;
(ii) φ *is topologically valid;*
(iii) φ *is true in any finite topological space;*
(iv) φ *is valid in* $\mathbb{R}^n$ *for* $n \geq 2$.

Proof. The equivalence of (*i*)-(*iii*) was proved by Artemov et. al. in (3). The equivalence of (*i*) and (*iv*) was proved by Duque in (10). This was a strengthening of a result proved by Slavnov in (38). □

Theorem 4.2.4. *(Incompleteness for* $\mathbb{R}$*) There exists* $\varphi \in L_{D,Q}$ *such that* φ *is valid in* $\mathbb{R}$, *but* φ *is not topologically valid.*

Proof. See (19) and (37). □

4.3 Kripke semantics for $S4C$

In this section we show that the logic $S4C$ can also be interpreted in the more familiar setting of Kripke frames. It is well known that the logic $S4$ (which does not include the 'temporal' modality Q) is interpreted in transitive, reflexive Kripke frames, and that such frames *just are* topological spaces of a certain kind. It follows that the Kripke semantics for $S4$ is just a special case of the topological semantics for $S4$. In this section, we show that the logic $S4C$ can be interpreted in transitive, reflexive Kripke frames with some additional 'dynamic' structure, and, again, that Kripke semantics for $S4C$ is a special case of the more general topological semantics for $S4C$. Henceforth, we assume that Kripke frames are both transitive and reflexive.

Defnition 4.3.1. *A* dynamic Kripke frame *is a triple* $\langle W, R, G\rangle$ *where* W *is a set,* R *is a refexive, transitive relation on* W*, and* $G : W \to W$ *is a function that is* R*-monotone in the following sense: for any* $u, v \in W$*, if* uRv*, then* $G(u)\ R\ G(v)$*.*

Defnition 4.3.2. *A* dynamic Kripke model *is a pair* $\langle F, V\rangle$ *where* $F = \langle W, R, G\rangle$ *is a dynamic Kripke frame and* $V : \mathrm{P} \to \wp(W)$ *is a valuation assigning to each propositional variable an arbitrary subset of* W*. We extend* V *to the set of all formulas in* $L_{\mathrm{D,Q}}$ *by the following recursive clauses:*

$V(\varphi \vee \psi) = V(\varphi) \cup V(\psi)$
$V(\neg\varphi) = W \setminus V(\varphi)$
$V(\mathrm{Q}\varphi) = G^{-1}(V(\varphi))$.
$V(\mathrm{D}\varphi) = \{w \in W \mid v \in V(\varphi)$ *for all* $v \in W$ *such that* $wRv\}$

Given a dynamic Kripke frame $K = \langle W, R, G\rangle$ we can impose a topology on W via the accessibility relation R. We defne the *open* subsets of W as those subsets that are upward closed under R:

(*) $O \subseteq W$ is open iff $x \in O$ and xRy implies $y \in O$

Recall that an *Alexandroff topology* is a topological space in which arbitrary intersections of open sets are open. The reader can verify that the collection of open subsets of W includes the entire space, the empty set, and is closed under arbitrary intersections and unions. Hence, viewing $\langle W, R\rangle$ as a topological space, the space is Alexandroff.

Going in the other direction, if X is an Alexandroff topology, we can defne a relation R on X by:

(@) xRy iff x is a point of closure of $\{y\}$

(Equivalently, y belongs to every open set containing x.) Clearly R is refexive. To see that R is transitive, suppose that xRy and yRz. Let O be an open set containing x. Then since x is a point of closure for $\{y\}$, $y \in O$. But since y is a point of closure for $\{z\}$, $z \in O$. So x is a point of closure for $\{z\}$ and xRz. So far, we have shown that *static* Kripke frames, $\langle W, R\rangle$ correspond to Alexandroff topologies. But what about the dynamical aspect? Here we invite the reader to verify that R-monotonicity of the function G is equivalent to continuity of G in the topological setting. It follows that *dynamic Kripke frames are just dynamic Alexandroff topologies*.

In view of the fact that every fnite topology is Alexandroff (if X is fnite, then there are only fnitely many open subsets of X), we have shown that fnite topologies are just fnite Kripke frames. This result, together with Theorem 4.2.3 (*iii*), gives the following completeness theorem for Kripke semantics:

Lemma 4.3.3. *For any formula $\varphi \in L_{\mathrm{D},\mathrm{Q}}$, the following are equivalent:*
(i) $S4C \vdash \varphi$;
(ii) φ is true in any fnite Kripke frame (= fnite topological space).

In what follows, it will be useful to consider not just arbitrary fnite Kripke frames, but frames that carry some additional structure. The notion we are after is that of a *stratifed dynamic Kripke frame*, introduced by Slavnov in (38). We recall his defnitions below.

Defnition 4.3.4. *Let $K = \langle W, R, G\rangle$ be a dynamic Kripke frame. A* cone *in K is any set $U_v = \{w \in W \mid vRw\}$ for some $v \in W$. We say that v is a* root *of U_v.*

Note in particular that any cone, U_v, in K is an open subset of W—indeed, the smallest open subset containing v.

Defnition 4.3.5. *Let $K = \langle U, R, G\rangle$ be a fnite dynamic Kripke frame. We say that K is* stratifed *if there is a sequence $\langle U_1, \ldots, U_n\rangle$ of pairwise disjoint cones in K with roots $u_1, \ldots, u_n$ respectively, such that $U = \bigcup_k U_k$, $G(u_k) = u_{k+1}$ for $k < n$, and G is injective. We say the stratifed Kripke frame has* depth n *and (with slight abuse of notation) we call u_1 the* root *of the stratifed frame.*

Note that it follows from R-monotonicity of G that $G(U_k) \subseteq U_{k+1}$, for $k < n$.

Defnition 4.3.6. *Defne the function CD ("circle depth") on the set of all formulas in $L_{\mathrm{D},\mathrm{Q}}$ inductively, as follows.*

$CD(p) = 0$ *for any propositional variable p;*
$CD(\varphi \vee \psi) = max\,\{CD(\varphi), CD(\psi)\}$;
$CD(\neg\varphi) = CD(\varphi)$;
$CD(\mathrm{D}\varphi) = CD(\varphi)$;
$CD(\mathrm{Q}\varphi) = 1 + CD(\varphi)$.

We also refer to $CD(\varphi)$ as the Q-depth of φ.

Lemma 4.3.7. *Suppose the formula φ is not a theorem of $S4C$, and $CD(\varphi) = n$. Then there is a stratifed fnite dynamic Kripke frame K with depth $n + 1$ such that φ is refuted at the root of K.*

Proof. The proof is by Lemma 4.3.3 and by a method of 'disjointizing' fnite Kripke frames. For the details, see (38). □

4.4 Algebraic semantics for $S4C$

We saw that the topological semantics for $S4C$ is a generalization of the Kripke semantics. Can we generalize further? Just as classical propositional logic is interpreted in Boolean algebras, we would like to interpret modal logics algebraically. Tarski and McKinsey showed that this can be done for the logic $S4$, interpreting the D-modality as an interior operator on a Boolean algebra. In this section we show that the same can be done for the logic $S4C$, interpreting the O-modality via O-operators on a Boolean algebra.

We denote the top and bottom elements of a Boolean algebra by 1 and 0, respectively.

Defnition 4.4.1. *A* topological Boolean algebra *is a Boolean algebra,* A*, together with an* interior operator I *on* A *that satisfes:*

(I_1) $I1 = 1$;
(I_2) $Ia \leq a$;
(I_3) $IIa = Ia$;
(I_4) $I(a \wedge b) = Ia \wedge Ib$.

Example 4.4.2. *The set of all subsets* $\mathcal{P}(X)$ *of a topological space* X *with set-theoretic meets, joins and complements and where the operator* I *is just the topological interior operator (for* $A \subseteq X$*,* $I(A) = Int(A)$*) is a topological Boolean algebra. More generally, any collection of subsets of* X *that is closed under fnite intersections, unions, complements and topological interiors is a topological Boolean algebra. We call any such algebra a* topological feld of sets.

Suppose A is a topological Boolean algebra with interior operator I. We defne the *open* elements in A as those elements for which

$$Ia = a \tag{4.1}$$

Defnition 4.4.3. *Let* A_1 *and* A_2 *be topological Boolean algebras. We say* $h : A_1 \to A_2$ *is a* Boolean homomorphism *if* h *preserves Boolean operations. We say* h *is a* Boolean embedding *if* h *is an injective Boolean homomorphism. We say* h *is a* homomorphism *if* h *preserves Boolean operations and the interior operator. We say* h *is an* embedding *if* h *is an injective homomorphism. Finally, we say* A_1 *and* A_2 *are* isomorphic *if there is an embedding from* A_1 *onto* A_2.

Defnition 4.4.4. *Let* A_1 *and* A_2 *be topological Boolean algebras, and let* $h : A_1 \to A_2$*. We say* h *is an* O-map *if*

(i) h *is a Boolean homomorphism*

(ii) *For any c open in A_1, $h(c)$ is open in A_2.*

An O-operator *is an O-map from a topological Boolean algebra to itself.*

Lemma 4.4.5. *Let A_1 and A_2 be topological Boolean algebras, with interior operators I_1 and I_2 respectively. Suppose that $h : A_1 \to A_2$ is a Boolean homomorphism. Then h is an O-map iff for every $a \in A_1$,*

$$h(I_1 a) \leq I_2(h(a)) \tag{4.2}$$

Proof. We let G_1 and G_2 denote the collection of open elements in A_1 and A_2 respectively. ($\Rightarrow$) Suppose h is an O-map. Then $h(I_1 a) \in \mathsf{G}_2$ by Defnition 4.4.4 (*ii*). Also, $I_1 a \leq a$, so $h(I_1 a) \leq h(a)$ (h is a Boolean homomorphism, hence preserves order). Taking interiors on both sides, we have $h(I_1 a) = I_2(h(I_1 a)) \leq I_2(ha)$. ($\Leftarrow$) Suppose that for every $a \in A_1$, $h(I_1 a) \leq I_2(h(a))$. Let $c \in \mathsf{G}_1$. Then $c = I_1 c$, so $h(c) = h(I_1 c) \leq I_2(h(c))$. But also, $I_2(h(c)) \leq h(c)$. So $h(c) = I_2(h(c))$ and $h(c) \in \mathsf{G}_2$. □

We are now in a position to state the algebraic semantics for the language $L_{\mathrm{D},\mathsf{Q}}$.

Defnition 4.4.6. *A* dynamic algebra *is a pair $\langle A, h\rangle$, where A is a topological Boolean algebra and h is an O-operator on A. A* dynamic algebraic model *is an ordered triple, $\langle A, h, V\rangle$, where A is a topological Boolean algebra, h is an O-operator on A, and $V : \mathsf{P} \to A$ is a valuation, assigning to each propositional variable $p \in \mathsf{P}$ an element of A. We say $\langle A, h, V\rangle$ is a model over A. We can extend V to the set of all formulas in $L_{\mathrm{D},\mathsf{Q}}$ by the following recursive clauses:*

$$V(\varphi \vee \psi) = V(\varphi) \vee V(\psi)$$
$$V(\neg\varphi) = -V(\varphi)$$
$$V(\mathrm{D}\varphi) = IV(\varphi)$$
$$V(\mathsf{Q}\varphi) = hV(\varphi)$$

(The remaining binary connectives, $\rightarrow$ and , and unary operator, $\Diamond$, are defned in terms of the above in the usual way.)

We defne standard validity relations. Let $N = \langle A, h, V\rangle$ be a dynamic algebraic model. We say φ is *true* in N ($N \models \varphi$) iff $V(\varphi) = 1$. Otherwise, we say φ is *refuted* in N. We say φ is *valid* in A ($\models_A \varphi$) if for any algebraic model N over A, $N \models \varphi$. Finally, we let $DML_A = \{\varphi \mid \models_A \varphi\}$ (*i.e.*, the set of validities in A). In our terminology, soundness of $S4C$ for A is the claim: $S4C \subseteq DML_A$. Completeness of $S4C$ for A is the claim: $DML_A \subseteq S4C$.

Proposition 4.4.7. *(Soundness) Let A be a topological Boolean algebra. Then* $S4C \subseteq DML_A$.

Proof. We have to show that the $S4C$ axioms are valid in A and that the rules of inference preserve truth. To see that (A1) is valid, note that:

$$
\begin{aligned}
V(\mathrm{Q}(\varphi \vee \psi)) &= h(V(\varphi) \vee V(\psi)) \\
&= h(V(\varphi)) \vee h(V(\psi)) \quad (h \text{ a Boolean homomorphism}) \\
&= V(\mathrm{Q}\varphi \vee \mathrm{Q}\psi)
\end{aligned}
$$

Thus $V(\mathrm{Q}(\varphi \vee \psi) \leftrightarrow (\mathrm{Q}\varphi \vee \mathrm{Q}\psi)) = 1$. Validity of (A2) is proved similarly. For (A3), note that:

$$
\begin{aligned}
V(\mathrm{QD}\varphi) &= h(IV(\varphi)) \\
&\leq Ih(V(\varphi)) \qquad \text{(by Lemma 4.4.5)} \\
&= V(\mathrm{DQ}\varphi)
\end{aligned}
$$

So $V(\mathrm{QD}\varphi) \leq V(\mathrm{DQ}\varphi)$ and $V(\mathrm{QD}\varphi \to \mathrm{DQ}\varphi) = 1$. This takes care of the special Q modality axioms. The remaining axioms are valid by soundness of $S4$ for any topological Boolean algebra—see *e.g.*, (33). To see that necessitation for Q preserves validity, suppose that φ is valid in A (*i.e.*, for every algebraic model $N = \langle A, h, V \rangle$, we have $V(\varphi) = 1$). Then $V(\mathrm{Q}\varphi) = h(V(\varphi)) = h(1) = 1$, and $\mathrm{Q}\varphi$ is valid in A. □

4.5 Reduced measure algebras

We would like to interpret $S4C$ not just in arbitrary topological Boolean algebras, but in algebras carrying a probability measure—or 'measure algebras.' In this section we show how to construct such algebras from separable metric spaces together with a σ-fnite Borel measure (defned below).

Defnition 4.5.1. *Let A be a Boolean σ-algebra, and let μ be a non-negative function on A, with $\mu(0) = 0$. We say μ is a measure on A if for any countable collection $\{a_n\}$ of disjoint elements in A, $\mu(\bigvee_n a_n) = \sum_n \mu(a_n)$.*

If μ is a measure on A, we say μ is *positive* if 0 is the only element at which μ takes the value 0. We say μ is *σ-fnite* if 1 is the countable join of elements in A with fnite measure.[2] Finally, we say μ is *normalized* if $\mu(1) = 1$.

[2] *I.e.*, there is a countable collection of elements A_n in A such that $\bigvee_n A_n = 1$ and $\mu(A_n) < \infty$ for each $n \in \mathbb{N}$.

Defnition 4.5.2. *A* measure algebra *is a Boolean σ-algebra A together with a positive, σ-fnite measure μ on A.*

Lemma 4.5.3. *Let A be a Boolean σ-algebra and let μ be a σ-fnite measure on A. Then there is a normalized measure ν on A such that for all* $a \in A$, $\mu(a) = 0$ *iff* $\nu(a) = 0$.

Proof. Since μ is σ-fnite, there exists a countable collection $\{s_n \mid n \geq 1\} \subseteq A$ such that $\bigvee_{n \geq 1} s_n = 1$ and $\mu(s_n) < \infty$ for each $n \geq 1$. WLOG we can assume the s_n's are pairwise disjoint (*i.e.*, $s_n \wedge s_m = 0$ for $m \neq n$). For any $a \in A$, let

$$\nu(a) = \sum_{n \geq 1} 2^{-n} \frac{\mu(a \wedge s_n)}{\mu(s_n)}$$

The reader can verify that ν has the desired properties. □

In what follows, we show how to construct measure algebras from a topological space, X, together with a Borel measure on X. The relevant defnition is given below.

Defnition 4.5.4. *Let X be a topological space. We say that μ is a* Borel measure *on X if μ is a measure defned on the σ-algebra of Borel subsets of X.*[3]

Let X be a topological space, and let μ be a σ-fnte Borel measure on X. We let $Borel(X)$ denote the collection of Borel subsets of X and let $Null_\mu$ denote the collection of measure-zero Borel sets in X. Then $Borel(X)$ is a Boolean σ-algebra, and $Null_\mu$ is a σ-ideal in $Borel(X)$. We form the quotient algebra

$$\mathsf{M}_X^\mu = Borel(X)/Null_\mu$$

(Equivalently, we can defne the equivalence relation $\sim$ on Borel sets in X by $A \sim B$ iff $\mu(A \triangle B) = 0$, where $\triangle$ denotes symmetric difference. Then M_X^μ is the algebra of equivalence classes under $\sim$.) Boolean operations in M_X^μ are defned in the usual way in terms of underlying sets:

$$\begin{aligned} |A| \vee |B| &= |A \cup B| \\ |A| \wedge |B| &= |A \cap B| \\ \neg|A| &= |X - A| \end{aligned}$$

Lemma 4.5.5. *There is a unique measure ν on* M_X^μ *such that* $\nu|A| = \mu(A)$ *for all A in* $Borel(X)$*. Moreover, the measure ν is σ-fnite and positive.*

[3] *I.e.*, on the smallest σ-algebra containing all open subsets of *X*.

Proof. See (15, p. 79). □

It follows from Lemma 4.5.5 that M_X^μ is a measure algebra. We follow Halmos (15) in referring to any algebra of the form M_X^μ as a ***reduced measure algebra***.[4]

Lemma 4.5.6. *Let X be a topological space and let μ be a σ-fnite Borel measure on X. Then for any $|A|, |B| \in \mathsf{M}_X^\mu$, $|A| \leq |B|$ iff $A \subseteq B \cup N$ for some $N \in Null_\mu$.*

Proof. ($\Rightarrow$) If $|A| \leq |B|$, then $|A| \wedge |B| = |A|$, or equivalently $|A \cap B| = |A|$. This means that $(A \cap B) \triangle A \in Null_\mu$, so $A - B \in Null_\mu$. But $A \subseteq B \cup (A - B)$. ($\Leftarrow$) Suppose $A \subseteq B \cup N$ for some $N \in \mathrm{Null}_\mu$. Then $A \cap (B \cup N) = A$, and $|A| \wedge |B \cup N| = |A|$. But $|B \cup N| = |B|$, so $|A| \wedge |B| = |A|$, and $|A| \leq |B|$. □

For the remainder of this section, let X be a separable metric space, and let μ be a σ-fnite Borel measure on X. Where the intended measure is obvious, we will drop superscripts, writing M_X for M_X^μ.

So far we have seen only that M_X^μ is a Boolean algebra. In order to interpret the D-modality of $S4C$ in M_X^μ, we need to construct an interior operator on this algebra (thus transforming M_X^μ into a topological Boolean algebra). We do this via the topological structure of the underlying space, X. Let us say that an element $a \in \mathsf{M}_X^\mu$ is *open* if $a = |U|$ for some open set $U \subseteq X$. We denote the collection of open elements in M_X^μ by G_X^μ (or, dropping superscripts, G_X).

Proposition 4.5.7. G_X^μ *is closed under* (*i*) *fnite meets and* (*ii*) *arbitrary joins.*

Proof. (i) This follows from the fact that open sets in X are closed under fnite intersections. (ii) Let $\{a_i \mid i \in I\}$ be a collection of elements in G_X^μ. We need to show that $sup\{a_i \mid i \in I\}$ exists and is equal to some element in G_X^μ. Since X is separable, there exists a countable dense set D in X. Let B be the collection of open balls in X centered at points in D with rational radius. Then any open set in X can be written as a union of elements in B. Let S be the collection of elements $B \in \mathsf{B}$ such that $|B| \leq a_i$ for some $i \in I$. We claim that

$$sup\{a_i \mid i \in I\} = |\bigcup S|$$

First, we need to show that $|\bigcup S|$ is an upper bound on $\{a_i \mid i \in I\}$. For each $i \in I$, $a_i = |U_i|$ for some open set $U_i \subseteq X$. Since U_i is open, it can be written as

[4] In fact, Halmos allows as 'measure algebras' only algebras with a normalized measure. We relax this constraint here, in order to allow for the 'reduced measure algebra' generated by the entire real line together with the usual Lebesgue measure. This algebra is, of course, isomorphic to M_X^μ, where X is the real interval $[0, 1]$, and μ is the usual Lebesgue measure on X. This amendment was

suggested by the anonymous referee.

a union of elements in $\mathcal{B}$. Moreover, each of these elements is a member of $\bigcup S$ (if $B \in \mathcal{B}$ and $B \subseteq U_i$, then $|B| \le |U_i| = a_i$). So $U_i \subseteq \bigcup S$ and $a_i = |U_i| \le |\bigcup S|$.

For the reverse inequality ($\geq$) we need to show that if m is an upper bound on $\{a_i \mid i \in I\}$, then $|\bigcup S| \le m$. Let $m = |M|$. Note that S is countable (since $S \subseteq \mathcal{B}$ and $\mathcal{B}$ is countable). We can write $S = \{B_n \mid n \in \mathbb{N}\}$. Then for each $n \in \mathbb{N}$, there exists $i \in I$ such that $|B_n| \le a_i \le m$. By Lemma 4.5.5, $B_n \subseteq M \cup N_n$ for some $N_n \in Null_\mu$. Taking unions, $\bigcup_n B_n \subseteq M \cup \bigcup_n N_n$, and $\bigcup_n N_n \in Null_\mu$. By Lemma 4.5.5, $|\bigcup S| = |\bigcup_n B_n| \le m$. □

We can now defne an interior operator, I_X^μ, on $\mathbb{M}_X^\mu$ via the collection of open elements, $\mathbb{G}_X^\mu$. For any $a \in \mathbb{M}_X^\mu$, let

$$I_X^\mu a = sup\,\{c \in \mathbb{G}_X^\mu \mid c \le a\}$$

Lemma 4.5.8. *I_X^μ is an interior operator.*

Proof. For simplicity of notation, we let I denote I_X^μ and let $\mathbb{G}$ denote $\mathbb{G}_X^\mu$. Then (I_1) follows from the fact that $1 \in \mathbb{G}$. (I_2) follows from the fact that a is an upper bound on $\{c \in \mathbb{G} \mid c \le a\}$. For ($I_3$) note that by ($I_2$), we have $IIa \le Ia$. Moreover, if $c \in \mathbb{G}$ with $c \le a$, then $c \le Ia$ (since Ia is supremum of all such c). Thus $\bigvee\{c \in \mathbb{G} \mid c \le a\} \le \bigvee\{c \in \mathbb{G} \mid c \le Ia\}$ and $Ia \le IIa$. For (I_4) note that since $a \wedge b \le a$, we have $I(a \wedge b) \le Ia$. Similarly, $I(a \wedge b) \le Ib$, so $I(a \wedge b) \le Ia \wedge Ib$. For the reverse inequality, note that $Ia \wedge Ib \le a$ (since $Ia \le a$), and similarly $Ia \wedge Ib \le b$. So $Ia \wedge Ib \le a \wedge b$. Moreover, $Ia \wedge Ib \in \mathbb{G}$. It follows that $Ia \wedge Ib \le I(a \wedge b)$. □

Remark 4.5.9. *Is the interior operator I_X^μ non-trivial? (That is, does there exist $a \in \mathbb{M}_X^\mu$ such that $Ia \neq a$?) This depends on the space, X, and the measure, μ. If we let X be the real interval, $[0, 1]$, and let μ be the Lebesgue measure on Borel subsets of X, then the interior operator is non-trivial. For the proof, see (22). But suppose μ is a non-standard measure on the real interval, [0, 1], defned by:*

$$\mu(A) = \begin{cases} 1 & \text{if } \frac{1}{2} \in A \\ 0 & \text{otherwise} \end{cases}$$

Then $Borel([0, 1])/Null_\mu$ is the algebra 2, and both elements of this algebra are 'open.' So $Ia = a$ for each element a in the algebra.

Remark 4.5.10. *The operator I_X^μ does not coincide with taking topological interiors on underlying sets. More precisely, it is in general not the case that for $A \subseteq X$, $I_X^\mu(|A|) = |Int(A)|$, where '$Int(A)$' denotes the topological interior of*

A. Let X be the real interval $[0,1]$ with the usual topology, and let μ be Lebesgue measure restricted to measurable subsets of X. Consider the set $X - \mathbb{Q}$ and note that $|X - \mathbb{Q}| = |X|$ ($\mathbb{Q}$ is countable, hence has measure zero). We have: $I^\mu_X(|X - \mathbb{Q}|) = I^\mu_X(|X|) = I^\mu_X(1) = 1$. However, $|Int(X - \mathbb{Q})| = |\emptyset| = 0$.

Remark 4.5.11. *Note that an element $a \in \mathbb{M}_\mu$ is open just in case $I^\mu_X a = a$. Indeed, if a is open, then $a \in \{c \in \mathbb{G}^\mu_X \mid c \leq a\}$. So $a = sup\{c \in \mathbb{G}^\mu_X \mid c \leq a\} = I^\mu_X a$. Also, if $I^\mu_X a = a$, then a is the join of a collection of elements in $\mathbb{G}^\mu_X$, and so $a \in \mathbb{G}^\mu_X$. This shows that the defnition of 'open' elements given above fts with the defnition in (1).*

In what follows, it will sometimes be convenient to express the interior operator I^μ_X in terms of underlying open sets, as in the following Lemma:

Lemma 4.5.12. *Let $A \subseteq X$. Then $I^\mu_X(|A|) = |\bigcup\{O\ open \mid |O| \leq |A|\}|$*

Proof. By defnition of I^μ_X, $I^\mu_X(|A|) = sup\{c \in \mathbb{G}^\mu_X \mid c \leq |A|\}$. Let $\mathbb{B}$ and D be as in the proof of Proposition 4.5.7, and let S be the collection of elements $B \in \mathbb{B}$ such that $|B| \leq |A|$. Then by the proof of Proposition 4.5.7, $I^\mu_X(|A|) = |\bigcup S|$. But now $\bigcup S = \bigcup\{O\text{open} \mid |O| \leq |A|\}$. (This follows from the fact that any open set $O \subseteq X$ can be written as a union of elements in $\mathbb{B}$.) Thus, $I^\mu_X(|A|) = |\bigcup S| = |\bigcup\{O\text{open} \mid |O| \leq |A|\}|$. □

We have shown that $\mathbb{M}^\mu_X$ together with the operator I^μ_X is a topological Boolean algebra. Of course, for purposes of our semantics, we are interested in O-operators on $\mathbb{M}^\mu_X$. How do such maps arise? Unsurprisingly, a rich source of examples comes from continuous functions on the underlying topological space X. Let us spell this out more carefully.

Defnition 4.5.13. *Let X and Y be topological spaces and let μ and ν be Borel measures on X and Y respectively. We say $f : X \to Y$ is* measure-zero preserving *(MZP) if for any $A \subseteq Y$, $\nu(A) = 0$ implies $\mu(f^{-1}(A)) = 0$.*

Lemma 4.5.14. *Let X and Y be separable metric spaces, and let μ and ν be σ-fnite Borel measures on X and Y respectively. Suppose B is a Borel subset of X with $\mu(B) = \mu(X)$, and $f : B \to Y$ is measure-zero preserving and continuous. Defne $h^{|\cdot|}_f : \mathbb{M}^\nu_Y \to \mathbb{M}^\mu_X$ by*

$$h^{|\cdot|}_f(|A|) = |f^{-1}(A)|$$

Then $h^{|\cdot|}_f$ is an O-map. In particular, if $X = Y$, then $h^{|\cdot|}_f$ is an O-operator.

Proof. First, we must show that $h^{|\cdot|}_f$ is well-defned.[5]

Indeed, if $|A| = |B|$, then

[5]Note that by continuity of f, $f^{-1}(A)$ is a Borel set in B, hence also a Borel set in X.

$\nu(A \triangle B) = 0$. And since f is MZP, $\mu(f^{-1}(A) \triangle f^{-1}(B)) = \mu(f^{-1}(A \triangle B)) = 0$. So $f^{-1}(A) \sim f^{-1}(B)$. This shows that $h_f^{|\cdot|}|A|$ is independent of the choice of representative, A. Furthermore, it is clear that $h_f^{|\cdot|}$ is a Boolean homomorphism. To see that it is an O-map, we need only show that if $c \in G^\nu_Y$, $h_f^{|\cdot|}(c) \in G^\mu_X$. But if $c \in G^\nu_Y$ then $c = |U|$ for some open set $U \subseteq Y$. By continuity of f, $f^{-1}(U)$ is open in B. So $f^{-1}(U) = O \cap B$ for some O open in X. So $h_f^{|\cdot|}(c) = |f^{-1}(U)| = |O| \in G^\mu_X$. □

By the results of the previous section, we can now interpret the language of $S4C$ in reduced measure algebras. In particular, we say an algebraic model $\langle A, h, V\rangle$ is a *dynamic measure model* if $A = M^\mu_X$ for some separable metric space X and a σ-fnite Borel measure μ on X.

We are particularly interested in the reduced measure algebra generated by the real interval, [0,1], together with the usual Lebesgue measure.

Defnition 4.5.15. *(Lebesgue Measure Algebra) Let I be the real interval* $[0, 1]$ *and let λ denote Lebesgue measure restricted to the Borel subsets of I. The* Lebesgue measure algebra *is the algebra* M^λ_I.

Because of it's central importance, we denote the Lebesgue measure algebra without subscripts or superscripts, by M. Furthermore, we denote the collection of open elements in M by G and the interior operator on M by I.

As in Defnition 4.4.6, we let $DML_M = \{\varphi \mid \ \models_M \varphi\}$ (*i.e.*, the set of validities in M). In our terminology, soundness of $S4C$ for M is the claim: $S4C \subseteq DML_M$. Completeness of $S4C$ for M is the claim: $DML_M \subseteq S4C$.

Proposition 4.5.16. *(Soundness) $S4C \subseteq DML_M$.*

Proof. Immediate from Proposition 4.4.7. □

Remark 4.5.17. *The algebra M is isomorphic to the algebra $Leb([0, 1])/Null_\mu$ where $Leb([0, 1])$ is the σ-algebra of Lebesgue-measureable subsets of the real interval* $[0, 1]$*, and $Null_\mu$ is the σ-ideal of Lebesgue measure-zero sets. This follows from the fact that every Lebesgue-measureable set in* $[0, 1]$ *differs from some Borel set by a set of measure zero.*

4.6 Isomorphisms between reduced measure algebras

In this section we use a well-known result of Oxtoby's to show that any reduced measure algebra generated by a separable metric space with a σ-fnite, nonatomic

Borel measure is isomorphic to $\mathbb{M}$. By Oxtoby's result, we can think of as the *canonical* separable measure algebra.

In the remainder of this section, let $\mathbb{J}$ denote the space $[0, 1] \setminus \mathbb{Q}$ (with the usual metric topology), and let δ denote Lebesgue measure restricted to the Borel subsets of $\mathbb{J}$.

Defnition 4.6.1. *A topological space X is* topologically complete *if X is homeomorphic to a complete metric space.*

Defnition 4.6.2. *Let X be a topological space. A Borel measure μ on X is* nonatomic *if $\mu(\{x\}) = 0$ for each $x \in X$.*

Theorem 4.6.3. *(Oxtoby, 1970) Let X be a topologically complete, separable metric space, and let μ be a normalized, nonatomic Borel measure on X. Then there exists a Borel set $B \subseteq X$ and a function $f : B \to \mathbb{J}$ such that $\mu(X \setminus B) = 0$ and f is a measure-preserving homeomorphism (where the measure on $\mathbb{J}$ is δ).*

Proof. See (30). □

Lemma 4.6.4. [6] *Suppose X and Y are separable metric spaces, and μ and ν are normalized Borel measures on X and Y respectively. If $f : X \to Y$ is a measure preserving homoemorphism, then M^{μ}_X is isomorphic to M^{ν}_Y.*

Proof. For simplicity of notation, we drop superscripts, writing simply M_X, G_X, and I_X, *etc.* Let $h_f^{|\cdot|} : \mathsf{M}_Y \to \mathsf{M}_X$ be defned by $h_f^{|\cdot|}(|A|) = |f^{-1}(A)|$. This function is well-defned because f is MZP and continuous. (The frst property ensures that $h_f^{|\cdot|}(|A|)$ is independent of representative A; the second ensures that $f^{-1}(A)$ is Borel.) Clearly $h_f^{|\cdot|}$ is a Boolean homomorphism. We can defne the mapping $h_{f^{-1}}^{|\cdot|} : \mathsf{M}_X \to \mathsf{M}_Y$ by $h_{f^{-1}}^{|\cdot|}(|A|) = |f(A)|$. Then $h_f^{|\cdot|}$ and $h_{f^{-1}}^{|\cdot|}$ are inverses, so $h_f^{|\cdot|}$ is bijective. We need to show that $h_f^{|\cdot|}$ preserves interiors—*i.e.*, $h_f^{|\cdot|}(I_Y a) = I_X h_f^{|\cdot|}(a)$. The inequality ($\leq$) follows from the fact that $h_f^{|\cdot|}$ is an O-map (see Lemma 4.5.14). For the reverse inequality, we need to see that $h_f^{|\cdot|}(I_Y a)$ is an upper bound on $\{c \in \mathsf{G}_X \mid c \leq h_f^{|\cdot|}(a)\}$. If $c \in \mathsf{G}_X$, then $h_{f^{-1}}^{|\cdot|}(c) \in \mathsf{G}_Y$ and if $c \leq h_f^{|\cdot|}(a)$, then $h_{f^{-1}}^{|\cdot|}(c) \leq h_{f^{-1}}^{|\cdot|}(h_f^{|\cdot|}(a)) = a$. Thus $h_{f^{-1}}^{|\cdot|}(c) \leq I_Y a$, and $c = h_f^{|\cdot|}(h_{f^{-1}}^{|\cdot|}(c)) \leq h_f^{|\cdot|}(I_Y a)$. □

[6]We can relax the conditions of the lemma, so that instead of requiring that f is measure-preserving, we require only that $\nu(f(S)) = 0$ iff $\mu(A) = 0$. In fact, we can further relax these conditions so that $f : B \to C$, where $B \subseteq X$, $C \subseteq Y$, $\mu(B \triangle X) = 0$, and $\nu(C \triangle Y) = 0$. We prove the lemma as stated because only this weaker claim is needed for the proof of Corollary 4.6.5.

Corollary 4.6.5. *Let X be a separable metric space, and let μ be a nonatomic σ-fnite Borel measure on X with $\mu(X) > 0$. Then,*

$$\mathbf{M}^{\mu}_{X} \cong \mathbf{M}$$

Proof. By Lemma 4.5.3, we can assume that μ is normalized.[7] Let X_{comp} be the completion of the metric space X. Clearly X_{comp} is separable. We can extend the Borel measure μ on X to a Borel measure μ^* on X_{comp} by letting $\mu^*(S) = \mu(S \cap X)$ for any Borel set S in X_{comp}. The reader can convince himself that μ^* is a normalized, nonatomic, σ-fnite Borel measure on X_{comp}, and that $\mathbf{M}^{\mu^*}_{X_{comp}} \cong \mathbf{M}^{\mu}_{X}$. By Theorem 4.6.3, there exists a set $B \subseteq X_{comp}$ and a function $f : B \to \mathbf{J}$ such that $\mu^*(B) = 1$ and f is a measure-preserving homeomorphism. By Lemma 4.6.4, $\mathbf{M}_{\mathbf{J}} \cong \mathbf{M}_{B}$. We have:

$$\mathbf{M} \cong \mathbf{M}_{\mathbf{J}} \cong \mathbf{M}_{B} \cong M^{\mu^*}_{X_{comp}} \cong \mathbf{M}^{\mu}_{X}$$

□

4.7 Invariance maps

At this point, we have at our disposal two key results: completeness of $S4C$ for fnite stratifed Kripke frames, and the isomorphism between $\mathbf{M}^{\mu}_{X}$ and $\mathbf{M}$ for any separable metric space X and σ-fnite, nonatomic Borel measure μ. Our aim in what follows will be to transfer completeness from fnite stratifed Kripke frames to the Lebesgue measure algebra, $\mathbf{M}$. But how to do this?

We can view any topological space as a topological Boolean algebra—indeed, as the topological feld of all subsets of the space (see Example 4.4.2). Viewing the fnite stratifed Kripke frames in this way, what we need is 'truth-preserving' maps between the algebras generated by Kripke frames and $\mathbf{M}^{\mu}_{X}$, for appropriately chosen X and μ. The key notion here is that of a "dynamic embedding" (defned below) of one dynamic algebra into another. Although our specifc aim is to transfer truth from Kripke algebras to reduced measure algebras, the results we present here are more general and concern truth preserving maps between arbitrary dynamic algebras.

Recall that a *dynamic algebra* is a pair $\langle A, h \rangle$ where A is a topological Boolean algebra, and h is an O-operator on A.

[7] More explicitly: If μ is σ-fnite, then by Lemma 4.5.3 there is a normalized Borel measure μ^* on X such that $\mu^*(S) = 0$ iff $\mu(S) = 0$ for each $S \subseteq X$. It follows that $\mathbf{M}^{\mu}_{X} \cong \mathbf{M}^{\mu^*}_{X}$ (where the isomorphism is not, in general, measure-preserving).

Defnition 4.7.1. *Let* $M_1 = \langle A_1, h_1\rangle$ *and* $M_2 = \langle A_2, h_2\rangle$ *be two dynamic algebras. We say a function* $h : M_1 \to M_2$ *is a* dynamic embedding *if*

(i) h *is an embedding of* A_1 *into* A_2*;*

(ii) $h \circ h_1 = h_2 \circ h$.

Lemma 4.7.2. *Let* $M_1 = \langle A_1, h_1, V_1\rangle$ *and* $M_2 = \langle A_2, h_2, V_2\rangle$ *be two dynamic algebraic models. Suppose that* $h : \langle A_1, h_1\rangle \to \langle A_2, h_2\rangle$ *is a dynamic embedding, and for every propositional variable* p*,*

$$V_2(p) = h \circ V_1(p)$$

Then for any $\varphi \in L_{\mathrm{D,Q}}$*,*

$$V_2(\varphi) = h \circ V_1(\varphi)$$

Proof. By induction on the complexity of φ. □

Corollary 4.7.3. *Let* $M_1 = \langle A_1, h_1, V_1\rangle$ *and* $M_2 = \langle A_2, h_2, V_2\rangle$ *be two dynamic algebraic models. Suppose that* $h : \langle A_1, h_1\rangle \to \langle A_2, h_2\rangle$ *is a dynamic embedding, and for every propositional variable* p*,*

$$V_2(p) = h \circ V_1(p)$$

Then for any $\varphi \in L_{\mathrm{D,Q}}$*,*

$$M_1 \models \varphi \ \textit{ iff } \ M_2 \models \varphi$$

Proof. $M_2 \models \varphi$ iff $V_2(\varphi) = 1$
iff $h \circ V_1(\varphi) = 1$ (by Lemma 4.7.2)
iff $V_1 = 1$ (since h is an embedding)

□

Let $\langle X, F\rangle$ be a dynamic topological space and let A_X be the topological feld of all subsets of X (see Example 4.4.2). We defne the function h_F on A_X by

$$h_F(S) = F^{-1}(S)$$

It is not diffcult to see that h_F is an O-operator. We say that $\langle A_X, h_F\rangle$ is the dynamic algebra *generated by* (or *corresponding to*) to the dynamic topological space $\langle X, F\rangle$.

Our goal is to embed the dynamic algebras generated by fnite dynamic Kripke frames into a dynamic measure algebra $\langle M^{\mu}_X, h\rangle$, where X is some appropriately chosen separable metric space and μ is a nonatomic, σ-fnite Borel measure on X.

In view of Corollary 4.7.3 and completeness for finite dynamic Kripke frames, this will give us completeness for the measure semantics. The basic idea is to construct such embeddings via 'nice' maps on the underlying topological spaces. To this end, we introduce the following new definition:

Definition 4.7.4. *Suppose X and Y are a topological spaces, and μ is a Borel measure on X. Let $\gamma : X \to Y$. We say γ has the* M-property *with respect to μ if for any subset $S \subseteq Y$:*

(i) $\gamma^{-1}(S)$ *is Borel;*

(ii) *for any open set $O \subseteq X$, if $\gamma^{-1}(S) \cap O \neq \emptyset$ then $\mu(\gamma^{-1}(S) \cap O) > 0$.*

Lemma 4.7.5. *Suppose $\langle X, F\rangle$ is a dynamic topological space, where X is a separable metric space, F is measure-zero preserving, and let μ be a σ-finite Borel measure on X with $\mu(X) > 0$. Suppose $\langle Y, G\rangle$ is a dynamic topological space, and $\langle \mathcal{A}_Y, h_G\rangle$ is the corresponding dynamic algebra. Let B be a subset of X with $\mu(B) = \mu(X)$, and suppose we have a map $\gamma : B \to Y$ that satisfies:*

(i) *γ is continuous, open and surjective;*

(ii) *$\gamma \circ F = G \circ \gamma$;*

(iii) *γ has the M-property with respect to μ.*

Then the map $\Phi : \langle \mathcal{A}_Y, h_G\rangle \to \langle \mathbf{M}_X^\mu, h_F\rangle$ defined by

$$\Phi(S) = |\gamma^{-1}(S)|$$

is a dynamic embedding.

Proof. By the fact that $\mathbf{M}_X^\mu$ is isomorphic to $\mathbf{M}_B$, we can view Φ as a map from $\langle \mathcal{A}_Y, h_G\rangle$ into $\langle \mathbf{M}_B^\mu, h_F^{|\cdot|}\rangle$ where h_F^μ is viewed as an operator on $\mathbf{M}_B$. Note that Φ is well-defined by the fact that γ satisfies clause (i) of the M-property. We need to show that (i) Φ is an embedding of $\langle \mathcal{A}_Y, h_G\rangle$ into $\langle \mathbf{M}_B^\mu, h_F\rangle$, and ($ii$) $\Phi \circ h_G = h_F^{|\cdot|} \circ \Phi$.

(i) Clearly Φ is a Boolean homomorphism. We prove that Φ is injective and preserves interiors.

- (Injectivity) Suppose $\Phi(S_1) = \Phi(S_2)$ and $S_1 \neq S_2$. Then $\gamma^{-1}(S_1) \sim \gamma^{-1}(S_2)$, and $S_1 \triangle S_2 \neq \emptyset$. Let $y \in S_1 \triangle S_2$. By surjectivity of γ, we have $\gamma^{-1}(y) \neq \emptyset$. Moreover, $\mu(\gamma^{-1}(y)) > 0$ (since γ has the M-property w.r.t. μ, and the entire space B is open). So $\mu(\gamma^{-1}(S_1) \triangle \gamma^{-1}(S_2)) = \mu(\gamma^{-1}(S_1 \triangle S_2)) \geq \mu(\gamma^{-1}(y)) > 0$. And $\gamma^{-1}(S_1) \not\sim \gamma^{-1}(S_2)$. $\perp$.
- (Preservation of Interiors) For clarity, we will denote the topological interior in the spaces Y and B by Int_Y and Int_B respectively, and the interior operator on M^μ_B by I. Let $S \subseteq Y$. It follows from continuity and openness of $\gamma : B \to Y$, that

  $$\gamma^{-1}(Int_Y(S)) = Int_B(\gamma^{-1}(S))$$

  Note that,

  – $\Phi(Int_Y(S)) = |\gamma^{-1}(Int_Y(S))|$
  $= |Int_B(\gamma^{-1}(S))|$
  $= |\bigcup\{O \text{ open in } B \mid O \subseteq \gamma^{-1}(S)\}|$

  – $I(\Phi(S)) = I|\gamma^{-1}(S)|$
  $= |\bigcup\{O \text{ open in } B \mid |O| \leq |\gamma^{-1}(S)|\}|$ (Lemma 4.5.12)

  Thus it is suffcient to show that for any open set $O \subseteq B$,

  $$O \subseteq \gamma^{-1}(S) \text{ iff } |O| \leq |\gamma^{-1}(S)|$$

  The left-to-right direction is obvious. For the right-to-left direction, suppose (toward contradiction) that $|O| \leq |\gamma^{-1}(S)|$ but that $O \not\subseteq \gamma^{-1}(S)$. Then $O \subseteq \gamma^{-1}(S) \cup N$ for some $N \subseteq B$ with $\mu(N) = 0$. Moreover, since $O \not\subseteq \gamma^{-1}(S)$, there exists $x \in O$ such that $x \notin \gamma^{-1}(S)$. Let $y = \gamma(x)$. Then $\gamma^{-1}(y) \cap O \neq \emptyset$. Since γ has the M-property with respect to μ, it follows that $\mu(\gamma^{-1}(y) \cap O) > 0$. But $\gamma^{-1}(y) \cap O \subseteq N$ (since $\gamma^{-1}(y) \cap O \subseteq O \subseteq \gamma^{-1}(S) \cup N$, and $\gamma^{-1}(y) \cap \gamma^{-1}(S) = \emptyset$). $\perp$.

We've shown that Φ is an embedding of $\langle A_Y, h_G \rangle$ into $\langle M^\mu_B, h_F \rangle$. In view of the isomorphism between M^μ_B and M_X, we have shown that Φ is an embedding of $\langle A_Y, h_G \rangle$ into M_X.

(ii) We know that $\gamma \circ F = G \circ \gamma$. Taking inverses, we have $F^{-1} \circ \gamma^{-1} = \gamma^{-1} \circ G^{-1}$. Now let $S \subseteq Y$. Then:

$$\begin{aligned}\Phi \circ h_G(S) &= |\gamma^{-1}(G^{-1}(S))| \\ &= |F^{-1}(\gamma^{-1}(S))| \\ &= h_F^{|\cdot|} \circ \Phi(S)\end{aligned}$$

□

4.8 Completeness of $S4C$ for the Lebesgue measure algebra with O-operators

In this section we prove the main result of the chapter: completeness of $S4C$ for the Lebesgue measure algebra, $\mathbb{M}$. Recall that completeness is the claim that $DML_{\mathbb{M}} \subseteq S4C$. In fact, we prove the contrapositive: For any formula $\varphi \in L_{\mathrm{D},\mathbb{Q}}$, if $\varphi \notin S4C$, then $\varphi \notin DML_{\mathbb{M}}$. Our strategy is as follows. If φ is a non-theorem of $S4C$, then by Lemma 4.3.7, φ is refuted in some finite stratified Kripke frame $K = \langle W, R, G\rangle$. Viewing the frame algebraically (*i.e.*, as a topological field of sets), we must construct a dynamic embedding $\Phi : \langle A_W, h_G\rangle \to \langle \mathbb{M}, h\rangle$, where $\langle A_W, h_G\rangle$ is the dynamic Kripke algebra generated by the dynamic Kripke frame K, and h is some O-operator on $\mathbb{M}$. In view of the isomorphism between $\mathbb{M}$ and $\mathbb{M}^{\mu}_X$ for any separable metric space, X, and nonatomic, σ-finite Borel measure μ on X with $\mu(X) > 0$, it is enough to construct a dynamic embedding of the Kripke algebra into $\mathbb{M}^{\mu}_X$ for appropriately chosen X and μ.

The constructions in this section are a modification of the constructions introduced in (38), where it is proved that $S4C$ is complete for topological models in Euclidiean spaces of arbitrarily large finite dimension. The modifications we make are measure-theoretic, and are needed to accommodate the new 'probabilistic' setting. We are very much indebted to Slavnov for his pioneering work in (38).[8]

4.8.1 Outline of the proof

Let us spell out the plan for the proof a little more carefully. The needed ingredients are all set out in Lemma 4.7.5. Our first step will be to construct the dynamic topological space $\langle X, F\rangle$ where X is a separable metric space, and F is a measure-zero preserving, continuous function on X. We must also construct a measure μ on the Borel sets of X that is nonatomic and σ-finite, such that $\mu(X) > 0$. We want to embed the Kripke algebra $\langle A_W, h_G\rangle$ into $\langle \mathbb{M}^{\mu}_X, h^{|\cdot|}\rangle$, and to do this, we must construct a topological map $\gamma : B \to W$, where $B \subseteq X$ and $\mu(B) = 1$, and γ satisfies the requirements of Lemma 4.7.5. In particular, we must ensure that

[8] Where possible, we have preserved Slavnov's original notation in (38).

(i) γ is open, continuous and surjective, (ii) $\gamma \circ F = G\gamma$ and (iii) γ has the M-property with respect to μ.

In Section 8.2, we show how to construct the dynamic space $\langle X, F \rangle$, and the Borel measure μ on X. In Section 8.3, we construct the map $\gamma : X \to W$, and show that it has the desired properties.

4.8.2 The topological carrier of countermodels

Let

$$X_n = I^1 \sqcup \cdots \sqcup I^n$$

where I^k is the k-th dimensional unit cube and $\sqcup$ denotes disjoint union. We would like X_n to be a metric space, so we think of the cubes I^k as embedded in the space $\mathbb{R}^n$, and as lying at a certain fixed distance from one another. For simplicity of notation, we denote points in I^k by $(x_1, \ldots, x_k)$, and do not worry about how exactly these points are positioned in $\mathbb{R}^n$.

For each $k < n$, define the map $F_k : I^k \to I^{k+1}$ by $(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, \frac{1}{2})$. We let

$$F(x) = \begin{cases} F_k(x) & \text{if } x \in I_k, k < n \\ x & \text{if } x \in I_n \end{cases}$$

Clearly F is injective. For each $k \geq 2$ we choose a privileged "midsection" $D_k = [0,1]^{k-1} \times \{\frac{1}{2}\}$ of I_k. Thus, $f(I_k) = D_{k+1}$ for $k < n$.

The space X_n will be the carrier of our countermodels (we will choose n according to the $\Box$-depth of the formula which we are refuting, as explained in the next section). We define a non-standard measure, μ, on X_n. This somewhat unusual measure will allow us to transfer countermodels on Kripke frames back to the measure algebra, $\mathbb{M}_{X_n}$.

Let μ on I_1 be Lebesgue measure on $\mathbb{R}$ restricted to Borel subsets of I_1. Suppose we have defined μ on $I^1, \ldots, I^k$. For any Borel set B in I^{k+1}, let $B_1 = B \cap D_{k+1}$, and $B_2 = B \setminus D_{k+1}$. Then $B = B_1 \sqcup B_2$. We define

$$\mu(B) = \mu(F^{-1}(B_1)) + \lambda(B_2)$$

where λ is the usual Lebesgue measure in $\mathbb{R}^{k+1}$. Finally, for any Borel set $B \subseteq X_n$, we let $\mu(B) = \sum_{k=1}^{n} \mu(B \cap I^k)$

Note that $\mu(I^1) = 1$, and in general $\mu(I^{k+1}) = \mu(I^k) + 1$. Thus $\mu(X_n) = \mu(I^1 \sqcup \cdots \sqcup I^n) = \sum_1^n k = \frac{1}{2}(n^2 + n)$.

Lemma 4.8.1. *μ is a nonatomic, σ-finite Borel measure on X_n.*

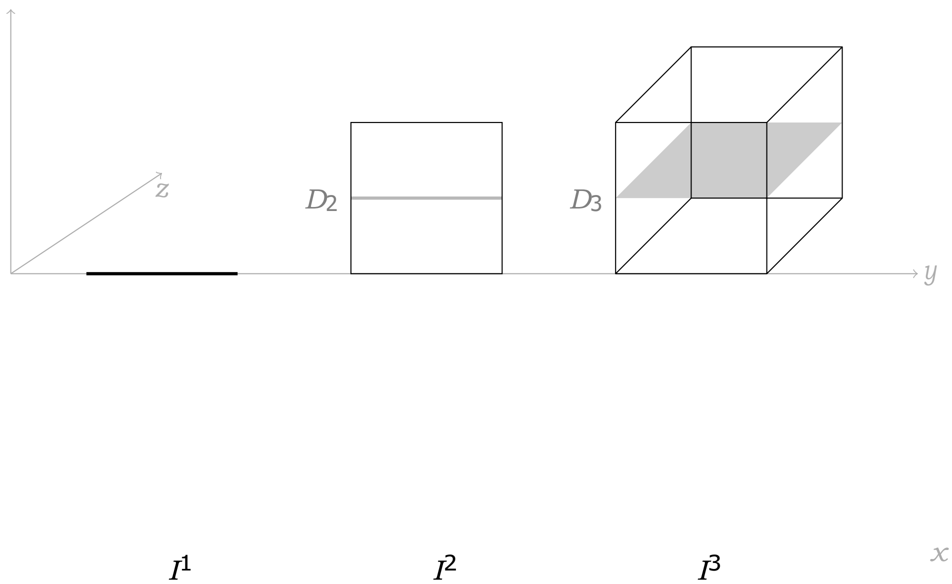

Figure 12: The space $X_3 = I^1 \sqcup I^2 \sqcup I^3$. Note that $\mu(I^1) = 1$, $\mu(I^2) = 2$, and $\mu(I^3) = 3$. The shaded regions in I^2 and I^3 denote the midsections, D_2 and D_3, respectively.

Proof. Clearly μ is nonatomic. Moreover, since $\mu(X_n) < \infty$, μ is σ-fnite. The only thing left to show is that μ is countably additive. Suppose that $\{B_m\}_{m\in\mathbb{N}}$ is a collection of pairwise disjoint subsets of X_n.

Claim 4.8.2. *For any $k \leq n$,*

$$\mu\Big(\bigcup_m (B_m \cap I^k)\Big) = \sum_m \mu(B_m \cap I^k)$$

(Proof of Claim: By induction on k.[9])

But now we have:

[9] The base case is by countable additivity of Lebesgue measure on the unit interval, $[0,1]$. For the induction step, suppose the claim is true for $k - 1$. Then we have:

$$\begin{aligned}
\mu\Big(\bigcup_m (B_m \cap I^k)\Big) &= \mu\Big[F^{-1}\Big(\bigcup_m (B_m \cap I^k \cap D^k)\Big)\Big] + \lambda\Big[\bigcup_m (B_m \cap I^k) \setminus D^k\Big] && \text{(Defn. of } \mu) \\
&= \mu\Big[\bigcup_m F^{-1}(B_m \cap I^k \cap D^k)\Big] + \sum_m \lambda((B_m \cap I^k) \setminus D^k) && \text{(Count. Add. of } \lambda) \\
&= \sum_m \mu[F^{-1}(B_m \cap I^k \cap D^k)] + \sum_m \lambda((B_m \cap I^k) \setminus D^k) && \text{(IH)} \\
&= \sum_m \mu[F^{-1}(B_m \cap I^k \cap D^k)] + \lambda((B_m \cap I^k) \setminus D^k)
\end{aligned}$$

$$= \;{}_{m}\,(B_m \cap I^k) \qquad \text{(Defn. of } \mu\text{)}$$

$$\mu\Big(\bigcup_m B_m\Big) = \sum_k \mu\Big[\Big(\bigcup_m B_m\Big) \cap I^k\Big] \qquad \text{(by defnition of } \mu)$$
$$= \sum_k \mu\Big[\bigcup_m (B_m \cap I^k)\Big]$$
$$= \sum_k \sum_m \mu(B_m \cap I^k) \qquad \text{(by Claim 4.8.2)}$$
$$= \sum_m \sum_k \mu(B_m \cap I^k)$$
$$= \sum_m \mu(B_m) \qquad \text{(by defnition of } \mu)$$

□

Lemma 4.8.3. *X is a separable metric space and $F : X_n \to X_n$ is measure-preserving and continuous.*

Proof. The set of rational points in I^k is dense in k ($k \leq n$), so X_n is separable. Continuity of F follows from the fact that F is a translation in $\mathbb{R}^n$; F is measure-preserving by the construction of μ. □

4.8.3 Completeness

Assume we are given a formula $\varphi \in L_{D,Q}$ such that φ is not a theorem of $S4C$ and let $n = CD(\varphi) + 1$. By Lemma 4.3.7, there is a fnite stratifed, dynamic Kripke model $K = \langle W, R, G, V \rangle$ of depth n such that φ is refuted at the root of K. In other words, there is a collection of pairwise disjoint cones $W_1, \ldots, W_n$ with roots $w_0^1, \ldots, w_0^n$ respectively, such that $W = \bigcup_{1 \leq k \leq} W_k$; G is injective; and $G(w_k) = w_{k+1}$ for each $k < n$; and $K, w_0 \models \varphi$. Let the space $X = X_n = I^1 \sqcup \ldots \sqcup I^n$ and the measure μ be as defned in the previous section. We construct a map $\tilde{\gamma} : X \to W$ in a countable number of stages. To do this we will make crucial use of the notion of E-nets, defned below:

Defnition 4.8.4. *Given a metric space S and $E > 0$, a subset Ω of S is an E-*net *for S if for any $y \in S$, there exists $x \in \Omega$ such that $d(x, y) < E$ (where d denotes the distance function in S).*

Observe that if S is compact, then for any $E > 0$ there is a *fnite* E-net for S.

Basic Construction. Let $w^1_{\text{root}} = w^1_0$, and let $w_1, \ldots, w_{r_1}$ be the R-successors of w^1_{root}. At the frst stage, we select r_1 pairwise disjoint closed cubes $T_1, \ldots, T_{r_1}$ in I^1, making sure that their total measure adds up to no more than $(\frac{1}{2})^{0+2}$—that is, $\sum_{k \leq r_1} \mu(T_k) < \frac{1}{4}$. For each x in the interior of T_k we let $\tilde{\gamma}(x) = w_k$ ($k \leq r_1$).

With slight abuse of notation we put $\tilde{\gamma}(T_k) = w_k$. We refer to $T_1, \dots, T_{r_1}$ as *terminal cubes*, and we let $I^1_1 = I^1 \setminus \bigcup^{r_1} \mathrm{Int}\,(T_k)$.

At any subsequent stage, we assume we are given a set I^1_i that is equal to I^1 with a fnite number of open cubes removed from it. Thus I^1_i is a compact set. We fnd a $\frac{1}{2^i}$-net Ω_i for I^1 and for each point $y \in \Omega_i$, we choose r_1 pairwise disjoint closed cubes, $T^y_1, \dots, T^y_{r_1}$ in the $\frac{1}{2^i}$-neighborhood of y, putting $\tilde{\gamma}(T^y_k) = w_k$ (for $k \le r_1$, with the same meaning as above). Again, we refer to the T_k's as terminal cubes. Since Ω_i is fnite, we create only a fnite number of new terminal cubes at this stage, and we make sure to do this in such a way as to remove a total measure of no more than $(\frac{1}{2})^{i+2}$. We let I^1_{i+1} be the set I^1_i minus the interiors of the new terminal cubes.

After doing this countably many times, we are left with some points in I^1 that do not belong to the interior of any terminal cube. We call such points *exceptional points* and we put $\tilde{\gamma}(x) = w^1_{\mathrm{root}}$ for each exceptional point $x \in I^1$. This completes the defnition of $\tilde{\gamma}$ on I^1.

Now assume that we have already defned $\tilde{\gamma}$ on I^j. We let $w^{j+1}_{\mathrm{root}} = w^{j+1}_0$ and let $w_1, \dots, w_{r_{j+1}}$ be the R-successors of w^{j+1}_{root}. We defne $\tilde{\gamma}$ on I^{j+1} as follows. At frst we choose r_{j+1} closed cubes $T_1, \dots, T_{r_{j+1}}$ in I^{j+1}, putting $\tilde{\gamma}(T_k) = w_k$ (for $k \le r_{j+1}$). In choosing $T_1, \dots, T_{r_{j+1}}$, we make sure that these cubes are not only pairwise disjoint (as before) but also disjoint from the midsection D_{j+1}. Again, we also make sure to remove a total measure of no more than $(\frac{1}{2})^{0+2}\mu(I^{j+1})$. We let $I^{j+1}_1 = I^{j+1} \setminus \bigcup_{k=1}^{r_{j+1}} \mathrm{Int}(T_k)$.

At stage i, we assume we are given a set I^{j+1}_i equal to I^{j+1} minus the interiors of a fnite number of closed cubes. Thus I^{j+1}_i is compact, and we choose a fnite $\frac{1}{2^i}$-net Ω_i for I^{j+1}. For each $y \in \Omega_i$ we choose r_{j+1} closed terminal cubes $T_1, \dots, T_{r_{j+1}}$ in the $\frac{1}{2}$-neighborhood of y. We make sure that these cubes are not only pairwise disjoint, but disjoint from the midsection D_{j+1}. Since Ω_i is fnite, we add only fnitely many new terminal cubes in this way. It follows that there is an E-neighborhood of D_{j+1} that is disjoint from all the terminal cubes added up to this stage. Moreover, for each terminal cube T of I^j defned at the ith stage, $F(T) \subseteq D_{j+1}$, and we let T^0 be some closed cube in I^{j+1} containing $F(T)$ and of height at most E. To ensure that the equality $\tilde{\gamma} \circ F(x) = G \circ \tilde{\gamma}(x)$ holds for all points x belonging to the interior of terminal cubes of I^j, we put:

$$\tilde{\gamma}(T^0) = G \circ \tilde{\gamma}(T)$$

Finally, we have added only fnitely many terminal cubes at this stage, and we do so in such a way as to make sure that the total measure of these cubes is no more than $(\frac{1}{2})^{i+2}\mu(I^{j+1})$. We let I^{j+1}_{i+1} be the set I^{j+1}_i minus the new terminal cubes added at this stage.

We iterate this process countably many times, removing a countable number of terminal cubes from I^{j+1}. For all exceptional points x in I^{j+1} (*i.e.*, points that do not belong to the interior of any terminal cube defned at any stage) we put $\tilde{\gamma}(x) = w^{j+1}_{roo}$. Noting that exceptional points of I^j are pushed forward under F to exceptional points in I^{j+1}, we see that the equality $\tilde{\gamma} \circ F(x) = G \quad \tilde{\gamma}(x)$ holds for exceptional points as well.

This completes the construction of $\tilde{\gamma}$ on X. We pause now to prove two facts about the map $\tilde{\gamma}$ that will be of crucial importance in what follows.

Lemma 4.8.5. *Let $E(I^j)$ be the collection of all exceptional points in I^j for some $j \leq n$. Then $\mu(E(I^j)) \geq \frac{1}{2}\mu(I^j)$.*

Proof. At stage i of construction of $\tilde{\gamma}$ on I^j, we remove from I^j terminal cubes of total measure no more than $(\frac{1}{2})^{i+2}\mu(I^j)$. Thus over countably many stages we remove a total measure of no mor than $\mu(I^j) \sum_{i\geq 0}(\frac{1}{2})^{i+2} = \frac{1}{2}\mu(I^j)$. The remaining points in I^j are all exceptional, so $\mu(E(I^j)) \geq \mu(I^j) - \frac{1}{2}\mu(I^j) = \frac{1}{2}\mu(I^j)$. □

Lemma 4.8.6. *Let $x \in I^j$ be an exceptional point for some $j \quad n$. Then $\tilde{\gamma}(x) = w^j_0$, and for any $E > 0$ and any $w_k \in W_j$ there is a terminal cube T contained in the E-neighborhood of x with $\tilde{\gamma}(T) = w_k$.*

Proof. Since $x \in I^j$ is exceptional, it belongs to I^j_i for each $i \in \mathbb{N}$. We can pick i large enough so that $\frac{1}{2} < \frac{E}{2}$ But then in the notations above, there exists a point $y \in \Omega_i$ such that $d(x, y) < \frac{E}{2}$. The statement now follows from the Basic Construction, since for each $w_k \in W_j$ there is a terminal cube T_k in the $\frac{1}{2}$-neighborhood of y (and so also in the $\frac{E}{2}$-neighborhood of y) with $\tilde{\gamma}(T_k) = w_k$. □

Construction of the maps, γ_l. In the basic construction we defned a map $\tilde{\gamma} : X \quad W$ that we will use in order to construct a sequence of 'approximation' maps, $\gamma_1, \gamma_2, \gamma_3, \ldots ...,$ where $\gamma_1 = \tilde{\gamma}$. In the end, we will construct the needed map, γ, as the limit (appropriately defned) of these approximation maps. We begin by putting $\gamma_1 = \tilde{\gamma}$. The *terminal cubes of* γ_1 and the *exceptional points of* γ_1 are the terminal cubes and exceptional points of the Basic Construction. Note that each of $I^1, \ldots, I^n$ contains countably many terminal cubes of γ_1 together with exceptional points that don't belong to any terminal cube.

Assume that γ_l is defned and that for each terminal cube T of γ_l, all points in the interior of T are mapped by γ_l to a single element in W, which we denote by $\gamma_l(T)$. Moreover, assume that:

(i) $\gamma_l \circ F = G \circ \gamma_l$

(ii) for any terminal cube T of γ_l in I^j, F maps T into some terminal cube T^0 of γ_l in I^{j+1}, for $j < n$

where F is again the embedding $(x_1, \ldots, x_j) \mapsto (x_1, \ldots, x_j, \frac{1}{2})$.

We now define γ_{l+1} on the interiors of the terminal cubes of γ_l. In particular, for any terminal cube T of γ_l in I^1, let $T^1 = T$ and let T^{j+1} be the terminal cube of I^{j+1} containing $F(T^j)$, for $j < n$. Then we have a system $T^1, \ldots, T^n$ exactly like the system $I^1, \ldots, I^n$ in the Basic Construction. We define γ_{l+1} on the interiors of $T^1, \ldots, T^n$ in the same way as we defined $\tilde{\gamma}$ on $I^1, \ldots, I^n$, letting $w^j_{\mathrm{root}} = \gamma_l(T^j)$ and letting $w_1, \ldots, w_{r_j}$ be the R-successors of w^j_{root}. The only modification we need to make is a measure-theoretic one. In particular, in each of the terminal cubes T^j, we want to end up with a set of exceptional points that carries non-zero measure (this will be important for proving that the limit map we define, γ, has the M-property with respect to μ). To do this, assume γ_{l+1} has been defined on $T^1, \ldots, T^j$, and that for $k \leq j$, $\mu(E(T^k)) \geq \frac{1}{2}\mu(T^k)$, where $E(T^k)$ is the set of exceptional points in T^k. When we define γ_{l+1} on T^{j+1}, we make sure that at the first stage we remove terminal cubes with a total measure of no more than $\frac{1}{2}^{0+2}\mu(T^{j+1})$. At stage i where we are given T^{j+1}_i we remove terminal cubes with a total measure of no more than $(\frac{1}{2})^{i+2}\mu(T^{j+1})$. Again, this can be done because at each stage i we remove only a finite number of terminal cubes, so we can make the size of these cubes small enough to ensure we do not exceed the allocated measure. Thus, over countably many stages we remove from T^{j+1} a total measure of no more than $\mu(T^{j+1})\sum_{i\geq 0}(\frac{1}{2})^{i+2} = \frac{1}{2}\mu(T^{j+1})$. Letting $E(T^{j+1})$ be the set of exceptional points in T^{j+1}, we have $\mu(E(T^{j+1})) \geq \frac{1}{2}\mu(T^{j+1})$.

We do this for each terminal cube T of γ_l in I^1. Next we do the same for all the remaining terminal cubes T of γ_l in I^2 (*i.e.*, those terminal cubes in I^2 that are disjoint from D_2), and again, for all the remaining terminal cubes T of γ_l in I^3 (the terminal cubes in I^3 that are disjoint from D_3), *etc.* At the end of this process we have defined γ_{l+1} on the interior of each terminal cube of γ_l. For any point $x \in X$ that does not belong to the interior of any terminal cube of γ_l, we put $\gamma_{l+1}(x) = \gamma_l(x)$. The terminal cubes of γ_{l+1} are the terminal cubes of the Basic Construction applied to each of the terminal cubes of γ_l. The points in the interior of terminal cubes of γ_l that do not belong to the interior of any terminal cube of γ_{l+1} are the exceptional points of γ_{l+1}.

In view of the measure-theoretic modifications we made above, we have the following analog of Lemma 4.8.5:

Lemma 4.8.7. *Let $l \in \mathbb{N}$ and let T be any terminal cube of γ_l and $E(T)$ be the set of exceptional points of γ_{l+1} in T. Then*

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$$\mu(E(T)) \geq \frac{}{2}\,\mu(T)$$

Furthermore, the reader can convince himself that we have the following analog of Lemma 4.8.6 for the maps γ_l:

Lemma 4.8.8. *Let x be an exceptional point of γ_l and let $\gamma_l(x) = w$. Then for any $E > 0$ and any v such that wRv, there is a terminal cube T of γ_l contained in the E-neighborhood of x with $\gamma_l(T) = v$.*

Finally, note that if x is an exceptional point of γ_l for some l, then $\gamma_l(x) = \gamma_{l+k}(x)$ for any $k \in \mathbb{N}$. We let B denote the set of points that are exceptional for some γ_l, and defne the map $\gamma : B \to W$ as follows:

$$\gamma(x) = \lim_{l\to\infty} \gamma_l(x)$$

Lemma 4.8.9. $\mu(B) = \mu(X)$.

Proof. Let T_l be the set of all points that belong to some terminal cube of γ_l. Note that $T_l \supseteq T_{l+1}$ for $l \in \mathbb{N}$, and $\mu(T_1)$ is fnite. Thus $\mu(\bigcap_l T_l) = \lim_{l\to\infty} \mu(T_l) = 0$. (The limit value follows from Lemma 4.8.7.) Finally, note that $B = X - \bigcap_l T_l$. So B is Borel, and $\mu(B) = \mu(X) - \mu(\bigcap_l T_l) = \mu(X)$. □

We have constructed a map $\gamma : B \to W$ where $\mu(B) = \mu(X)$. Moreover, by the Basic Construction, we have $\gamma_l \circ F(x) = G \circ \gamma_l(x)$ for each $l \in \mathbb{N}$. It follows that $\gamma \circ F(x) = G \circ \gamma(x)$ for $x \in B$. All that is left to show is that (*i*) γ is continuous, open, and surjective; and (*ii*) γ has the M-property with respect to μ.

Lemma 4.8.10. *γ has the M-property with respect to μ.*

Proof. We show that for any subset $S \subseteq W$, (*i*) $\gamma^{-1}(S)$ is Borel; and (*ii*) for any open set $O \subseteq X$, if $\gamma^{-1}(S) \cap O \neq \emptyset$ then $\mu(\gamma^{-1}(S) \cap O) \neq 0$. Note that since W is fnite, it is suffcient to prove this for the case where $S = \{w\}$ for some $w \in W$.

(i) Note that $x \in \gamma^{-1}(w)$ iff x is exceptional for some γ_l and x belongs to some terminal cube T of γ_{l-1}, with $\gamma_{l-1}(T) = w$. There are only countably many such cubes, and the set of exceptional points in each such cube is closed. So $\gamma^{-1}(w)$ is a countable union of closed sets, hence Borel.

(ii) Suppose that O is open in X with $\gamma^{-1}(w) \cap O \neq \emptyset$. Let $x \in \gamma^{-1}(w) \cap O$. Again, x is exceptional for some γ_l. Pick $E > 0$ such that the E-neighborhood of x is contained in O. By Lemma 4.8.8, there is a terminal cube T of γ_l contained in the E-neighborhood of x such that $\gamma_l(T) = w$ (since wRw). Letting $E(T)$ be the set of exceptional points of γ_{l+1} in T, we know that $E(T) \subseteq \gamma^{-1}(w)$. But by Lemma 4.8.7, $\mu(E(T)) \geq \frac{1}{2}\mu(T) > 0$. So $E(T)$ is a subset of $\gamma^{-1}(w) \cap O$ of non-zero measure, and $\mu(\gamma^{-1}(w) \cap O) > 0$.

□

In what follows, for any $w \in W$, let $U_w = \{v \in W \mid wRv\}$ (*i.e.*, U_w is the smallest open set in W containing w).

Lemma 4.8.11. *γ is continuous.*

Proof. Let U be an open set in W and suppose that $x \in \gamma^{-1}(U)$. Let $\gamma(x) = w \in U$. Then x is exceptional for some γ_l. So x belongs to an (open) terminal cube T of γ_{l-1} with $\gamma_{l-1}(T) = w$. By R-monotonicity of $\langle \gamma_l(y) \rangle_n$ for all $y \in B$, we know that for any $y \in T$, $\gamma(y) \in U_w$—*i.e.*, $T \subseteq \gamma^{-1}(U_w)$. Moreover, since $w \in U$ and U is open, we have $U_w \subseteq U$. Thus $x \in T \subseteq \gamma^{-1}(U)$. This shows that $\gamma^{-1}(U)$ is open in X. □

Lemma 4.8.12. *γ is open.*

Proof. Let O be open in B and let $w \in \gamma(O)$. We show that $U_w \subseteq \gamma(O)$. We know that there exists $x \in O$ such that $\gamma(x) = w$. Moreover, x is exceptional for some γ_l. Pick $E > 0$ small enough so that the E-neighborhood of x is contained in O. By Lemma 4.8.8, for each $v \in U_w$ there is a terminal cube T_v of γ_l contained in the E-neighborhood of x such that $\gamma_l(T_v) = v$. But then for any exceptional point y_v of γ_{l+1} that lies in T_v, we have $\gamma(y_v) = \gamma_{l+1}(y_v) = v$, and $y_v \in O$. We have shown that for all $v \in U_w$, $v \in \gamma(O)$. It follows that $\gamma(O)$ is open. □

Lemma 4.8.13. *γ is surjective.*

Proof. This follows immediately from the fact that γ 'hits' each of the roots, $w_0^1, \dots, w_0^{n+1}$, of K and γ is open. □

Corollary 4.8.14. *φ is refuted in* M.

Proof. We stipulated that φ is refuted in the dynamic Kripke model $K = \langle W, R, G, V_1 \rangle$. Equivalently, letting $M_1 = \langle A_K, h_G, V_1 \rangle$ be the dynamic algebraic model corresponding to K, φ is refuted in M_1. By Lemma 4.8.11, Lemma 4.8.12, Lemma 4.8.13, and Lemma 4.8.10, we showed that $\gamma : X \to W$ is (i) continuous, open and surjective; (ii) $\gamma \circ f = G \circ \gamma$; and (iii) γ has the M-property with respect to μ. Thus by Lemma 4.7.5, the map $\Phi : \langle A_K, h_G \rangle \to \langle \mathsf{M}_X^\mu, h_F \rangle$ defined by

$$\Phi(S) = |\gamma^{-1}(S)|$$

is a dynamic embedding. We now define the valuation $V_2 : \mathsf{P} \to \mathsf{M}_X^\mu$ by:

$$V_2(p) = \Phi \circ V_1(p)$$

and we let $M_2 = \langle \mathsf{M}_X^\mu, h_F, V_2 \rangle$. By Corollary 4.7.3, $M_2 \models \varphi$. In view of the isomorphism $M_X^\mu \sim \mathsf{M}$, we have shown that φ is refuted in M. □

We have shown that for *any* formula $\varphi \notin S4C$, φ is refuted in $\mathbf{M}$. We conclude the section by stating this completeness result more formally as follows:

Theorem 4.8.15. *$DML_{\mathbf{M}} \subseteq S4C$.*

4.9 Completeness for a single measure model

In this section we prove a strengthening of the completeness result of the previous section, showing that there is a single dynamic measure model $\langle \mathbf{M}, h, V \rangle$ in which every non-theorem of $S4C$ is refuted.

Defnition 4.9.1. *Denote by $\mathbf{M}^{\omega}$ the product $\mathbf{M} \times \mathbf{M} \times \mathbf{M} \ldots$ This is a Boolean algebra, where Boolean operations are defned component-wise:*

$$(a_1, a_2, a_3, \ldots) \vee (b_1, b_2, b_3, \ldots) = (a_1 \vee b_1, a_2 \vee b_2, a_3 \vee b_3, \ldots)$$
$$(a_1, a_2, a_3, \ldots) \wedge (b_1, b_2, b_3, \ldots) = (a_1 \wedge b_1, a_2 \wedge b_2, a_3 \wedge b_3, \ldots)$$
$$-(a_1, a_2, a_3, \ldots) = (-a_1, -a_2, -a_3, \ldots)$$

Defnition 4.9.2. *We say $(a_1, a_2, a_3, \ldots)$ is an* open *element in $\mathbf{M}^{\omega}$ if a_k is open in $\mathbf{M}$ for each $k \in \mathbb{N}$.*

The collection of open elements in $\mathbf{M}^{\omega}$ is closed under fnite meets, arbitrary joins and contains the top and bottom element (since operations in $\mathbf{M}^{\omega}$ are componentwise). We defne the operator I_{ω} on $\mathbf{M}^{\omega}$ by:

$$I_{\omega}(a_1, a_2, a_3, \ldots) = (Ia_1, Ia_2, Ia_3, \ldots)$$

Then I_{ω} is an interior operator on $\mathbf{M}^{\omega}$ (the proof is the same as the proof of Lemma 4.5.8). So the algebra $\mathbf{M}^{\omega}$ together with the interior operator I_{ω} is a topological Boolean algebra.

Lemma 4.9.3. *There is a dynamic algebraic model $M = \langle \mathbf{M}^{\omega}, h, V \rangle$ such that for any formula $\varphi \in L_{D,Q}$, the following are equivalent:*

(i) $S4C \vdash \varphi$;

(ii) $M \models \varphi$.

Proof. Let φ_k be an enumeration of all non-theorems of $S4C$ (there are only countably many formulas, so only countably many non-theorems). By completeness of $S4C$ for $\mathbf{M}$, for each $k \in \mathbb{N}$, there is a model $M_k = \langle \mathbf{M}, h_k, V_k \rangle$ such that $M_k \models \varphi_k$. We construct a model $M = \langle \mathbf{M}^{\omega}, h, V \rangle$, where h and V are defned as follows. For any $\langle a_k \rangle_{k\in\mathbb{N}} = (a_1, a_2, a_3, \dots) \in \mathbf{M}^{\omega}$, and for any propositional variable p:

$$h((a_1, a_2, a_2, \dots)) = \langle h_k(a_k) \rangle_{k\in\mathbb{N}}$$

$$V(p) = \langle V_k(p) \rangle_{k\in\mathbb{N}}$$

(The fact that h is an O-operator follows from the fact that h is computed componentwise according to the h_k's, and each h_k is an O-operator).

We can now prove the lemma. The direction $(i) \Rightarrow (ii)$ follows from Proposition 4.4.7. We show $(ii) \Rightarrow (i)$, by proving the contrapositive. Suppose that $S4C \not\models \varphi$. Then $\varphi = \varphi_k$ for some $k \in \mathbb{N}$. We claim that

$$\pi_k V(\varphi) = V_k(\varphi)$$

where π_k is the projection onto the kth coordinate. (Proof: By induction on complexity of φ, and the fact that π_k is a topological homomorphism.) In particular, $\pi_k V(\varphi_k) = V_k(\varphi_k) \neq 1$. So $V(\varphi_k) = 1$, and $M \not\models \varphi_k$. □

Lemma 4.9.4. $\mathbf{M}^{\omega}$ *is isomorphic to* $\mathbf{M}$.

Proof. We need to construct an isomorphism from $\mathbf{M}^{\omega}$ onto $\mathbf{M}$. Let $(a_1, a_2, a_3, \dots)$ be an arbirary element in $\mathbf{M}^{\omega}$. Then for each $k \in \mathbb{N}$, we can choose a set $A_k \subseteq [0, 1]$ such that $a_k = |A_k|$ and $1 \notin A_k$. We defne a sequence of points s_k in the real interval [0,1] as follows:

$$s_0 = 0$$
$$s_1 = 1/2$$
$$s_2 = 3/4$$

In general, $s_k = \frac{2^k - 1}{2^k}$ $(k \geq 1)$. We now defne a sequence of intervals I_k having the a_k's as endpoints:

$$I_0 = [0, \tfrac{1}{2})$$
$$I_1 = [\tfrac{1}{2}, \tfrac{3}{4})$$
$$I_2 = [\tfrac{3}{4}, \tfrac{7}{8})$$

and in general $I_k = [s_k, s_{k+1})$. Our idea is to map each set A_k into the interval I_k. We do this by letting $B_k = l_k A_k + s_k$ where l_k is the length of I_k. Clearly $B_k \subseteq I_k$ and $B_k \cap B_j = \emptyset$ for all $k \neq j$. We can now defne the map $h : \mathbf{M}^{\omega} \to \mathbf{M}$ by:

$$h(a_1, a_2, a_3, \dots) = \left| \bigcup_{k\in\mathbb{N}} B_k \right|$$

where B_k is defned as above. The reader can now verify that h is an isomorphism. □

Corollary 4.9.5. *There is a dynamic measure model $M = \langle M, h, V\rangle$ such that for any formula $\varphi \in L_{D,Q}$, the following are equivalent:*

(i) $S4C \vdash \varphi$;

(ii) $M \models \varphi$.

Proof. Immediate from Lemma 4.9.3 and Lemma 4.9.4. □

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Appendix A

'Connected' and 'Limited' in Gunky Space

In (1), Arntzenius takes as topological primitives the relation of being 'connected' and the property of being 'limited.' (These frst appeared together in Roeper's axiomatization of what he called 'region-based topology.' [1]) Intuitively, two regions are connected if they overlap or at least share a boundary point; a region is limited if it is bounded from the outside. Arntzenius defnes these relations in reduced measure algebras by giving defnitions for pointy topological spaces that are invariant under differences of measure zero:

[1]Roeper's ten axioms for pointless topology are as follows:

(A_1) If pointless region A is connected to pointless region B, then B is connected to A.

(A_2) Every pointless region that is not the pointless 1null region' is connected to itself.

(A_3) The null region is not connected to any pointless region.

(A_4) If A is connected to B and B is a part of C then A is connected to C.

(A_5) If A is connected to the 'fusion' of B and C, then A is connected to B or A is connected to C.

(A_6) The null region is limited.

(A_7) If A is limited and B is a part of A then B is limited.

(A_8) If A and B are limited then the fusion of A and B is limited.

(A_9) If A is connected to B then there is a pointless limited region C such that C is a part of B, and A is connected to C.

(A_{10}) If A is limited, B is not the pointless null region, and A is not connected to the complement of B, then there is a pointless region C which is non-null and limited, such that A is not connected to the complement of C, and C is not connected to the complement of B.

Arntzenius shows that on his defnitions of 'connectedness' and 'limited' for elements of reduced measure algebras, axioms (A_1) - (A_9) are satisfed, but (A_{10}) fails.

Defnition A.1 (Arntzenius: Connected)**.** *Pointy Borel sets A and B are* connected *if there exists a point p such that any open set containing p has an intersection of non-zero measure with both A and B.*

Defnition A.2 (Arntzenius: Limited)**.** *Pointy Borel set A is* limited *just in case for some compact pointy set B we have measure $A \cap B$ = measure (A).*[2]

In this appendix, we suggest a way of reproducing these relations in the measure-theoretic setting by introducing a topological basis for the collection of open elements in the algebra.

Recall the notion of a *basis* in pointy topology.

Defnition A.3. *Let $\langle X, \mathcal{T} \rangle$ be a topological space. A subset $\mathcal{B}$ of $\mathcal{T}$ is a* basis *if every member of $\mathcal{T}$ is a union of members of $\mathcal{B}$.*

In the real line with its standard topology, for example, we could take as a basis the collection of all open intervals, or the collection of all rational open intervals (intervals with rational endpoints). Let us defne an analogous notion for reduced measure algebras.

Defnition A.4. *Let $\mathcal{M}$ be a reduced measure algebra, and let $\mathcal{G}$ be the corresponding collection of open elements. A subset $\mathcal{B}$ of $\mathcal{G}$ is a* basis *if every member of $\mathcal{G}$ is a join of members of $\mathcal{B}$.*

In the remainder of this appendix, let $\mathcal{M}$ denote a reduced measure algebra arising from n-dimensional Euclidean space together with standard Lebesgue measure. We select as our basis the collection of elements represented by n-dimensional open cubes, or the collection of elements represented by n-dimensional open spheres (alternatively, *rational* cubes and *rational* spheres).

We now defne the relations of connectedness and limitedness by reference to this basis.

Defnition A.5. *Let a and b be elements of $\mathcal{M}$. Then a and b are* connected *if there exists a set $\{c_n \mid n \in \mathbb{N}\}$ of non-zero basic open elements in $\mathcal{M}$ such that*

$$\lim_{n \to \infty} \textit{measure}\,(c_n) = 0$$

$$c_n > c_{n+1}$$

and

$$c_n \wedge a = 0, \; c_n \wedge b \neq 0$$

for all $n \in \mathbb{N}$.

[2] In fact, Arntzenius gives a different but equivalent formulation: A is *limited* if there is a compact pointy set B such that measure$(A \cap \text{complement}(B)) = 0$.

Defnition A.6. *Let a be an element of the Lebesgue measure algebra, M. Then a is* limited *if there exists a basic open element of the algebra, c, such that $a \leq c$.*

The following two propositions show that Defnitions A.1 and A.2 are equivalent to Defnitions A.5 and A.6, respectively.

Proposition A.7. *Two elements a and b of M are connected according to Defnition A.1 if and only if they are connected according to Defnition A.5.*

Proof. Let A and B be Borel subsets of the real line, and let a and b be the corresponding elements of M. If a and b are connected according to Defnition A.1, then there is a point, p, such that any open set containing p has an intersection of non-zero measure with A and with B. Let C_n be the open interval centered at p with length $(\frac{1}{2})^n$, and let c_n be the corresponding element of M ($n \in \mathsf{N}$). Then c_n is a descending chain of non-zero basic open elements such that $\lim_{n\to\infty} measure\,(c_n) = 0$. Moreover, $measure(c_n \wedge a) = measure(C_n \cap A) = 0$.[3] This shows that if two elements of the algebra are connected according to Defnition A.1, then they are also connected according to Defnition A.5.

For the converse, suppose that a and b are connected according to Defnition A.5. Then there is a descending sequence, $\langle c_n \rangle$, of non-zero basic open elements in M with measure tending to zero, such that c_n intersects both a and b for all $n \in \mathsf{N}$. Let C_n be a representative open interval of c_n. Then C_n has left and right endpoints, which we denote by L_n and R_n, respectively. Note that $\langle L_n \rangle$ and $\langle R_n \rangle$ are bounded, monotone sequences of real numbers, hence converge. Moreover, since the measure of c_n tends to zero, these sequences converge to the same point, which we denote by p. The reader can now convince herself that any open set of reals containing p has an intersection of non-zero measure with both A and B. This shows that if two elements of the algebra are connected according to Defnition A.5, then they are also connected according to Defnition A.1. □

Proposition A.8. *An element a of M is limited according to Defnition A.2 if and only if they it is limited according to Defnition A.6.*

Proof. Suppose that a is an element of M and that A is a representative of a satisfying Defnition A.2. Then there is a compact set B such that measure $(A \cap B) =$ measure (A). Since these sets live in n-dimensional Euclidean space (*i.e.*, R^n), B is closed and bounded. This means that there is a closed interval, C, such

[3] Here we use 'measure' both for the Lebesgue measure on the real line, and for the measure function on the Lebesgue measure algebra. Strictly speaking, these functions have different domains, and so should be denoted differently. We trust the sloppiness here will not lead to any obscurity.

that $B \subseteq C$. Let b and c be the elements of $\mathcal{M}$ corresponding to pointy sets B and C, respectively. Then c is a basic open element, and we have:

$$\text{measure}\,(a) = \text{measure}\,(a \wedge b) \leq \text{measure}\,(a \wedge c) \leq \text{measure}\,(a)$$

It follows that $a \leq c$. This shows that if an element of the algebra is limited according to Defnition A.2, then it is also limited according to Defnition A.6.

For the converse, suppose that a is limited according to Defnition A.6, and that A is a representative of a. Then there is a basic open element, b, such that $a \leq b$. Let B be an open interval representative of b. The closure of B, $Cl(B)$, is a compact pointy set. Moreover, measure $(A \cap B)$ = measure (A). This shows that if an element of the algebra is limited according to Defnition A.6, then it is also limited according to Defnition A.2. $\square$